TENSOR SPHERICAL HARMONICS

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FOREWORD

This work contains material prepared by Jon Mathews shortly before his death in a storm at sea in December, 1979. The material had been left in carefully typed form with the clear intent that it be useful for those working with tensor spherical harmonics. It is reproduced here exactly as he left it to fulfill this intent.

CONTENTS

1. Introduction

We present here a summary of the basic properties of scalar, vector and tensor spherical harmonics. The properties are presented without derivations or proofs, which can be carried out by a variety of methods, such as the angular momentum techniques of Racah and others - see Rose for examples - or simply by using identities based on associated Legendre polynomials.

In Section 5 we give tables of scalar, vector and tensor harmonics for $j \leq 4$.

We have found the following sources useful:

Abramowitz, M. and I. A. Stegun, Handbook of Mathematical Functions (U. S. Government Printing Office, 1965).

Jackson, J. D., Classical Electromagnetism (Wiley, 1965).

Morse, P. M. and M. Feshback, Methods of Theoretical Physics (McGraw-Hill, 1953).

Rose, M. E., Elementary Theory of Angular Momentum (Wiley, 1957).

2. Definitions and basic properties

2A. Scalar spherical harmonics $Y_{jm}(\Omega)$

These are just the usual spherical harmonics, defined, for example, in Jackson, eq. (3.53).

$$Y_{jm}(\Omega) = Y_{jm}(\theta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi} \; \frac{(\ell-m)!}{(\ell+m)!}} \; P_\ell^m (\cos\theta) \, e^{im\varphi} \tag{1}$$

where the associated Legendre function $P_\ell^m(\cos\theta)$ is defined by

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!} \left(1-x^2\right)^{\frac{m}{2}} \left(\frac{d}{dx}\right)^{\ell+m} \left(x^2-1\right)^\ell \qquad (-\ell \le m \le \ell)$$

or equivalently

$$P_\ell^m(x) = \begin{cases} (-1)^m \left(1-x^2\right)^{\frac{m}{2}} \left(\frac{d}{dx}\right)^m P_\ell(x) & (m \geq 0) \\ (-1)^m \frac{(\ell+m)!}{(\ell-m)!} P_\ell^{-m}(x) & (m < 0) \end{cases}$$

where $P_\ell(x)$ is the usual Legendre polynomial:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{d}{dx}\right)^\ell \left(x^2-1\right)^\ell$$

The Y_{jm} are orthogonal functions; the normalization factor in (1) is chosen so that they are ortho<u>normal</u>:

$$\int d\Omega \; Y^*_{jm}(\Omega) \, Y_{j'm'}(\Omega) = \delta_{jj'} \, \delta_{mm'} ,$$

$$(d\Omega = \sin\theta \, d\theta \, d\varphi)$$

The parity of Y_{jm} is $(-1)^j$:

$$P \, Y_{jm}(\Omega) = Y_{jm}(\pi-\theta, \pi+\varphi) = (-1)^j \, Y_{jm}(\Omega)$$

Note that the function $\Psi(\vec{x}) = f_j(kr)Y_{jm}(\Omega)$ is a solution of the Helmholtz equation

$$\left(\nabla^2 + k^2\right) \Psi \left(\vec{x}\right) = 0 \tag{2}$$

provided $f_j(x)$ is a spherical Bessel function:

$$f_j(x) = j_j(x), \text{ or } n_j(x), \text{ or } h_j^{(1)}(x), \text{ or any linear combination of these with constant coefficients} \tag{3}$$

We shall use the notation $f_j(x)$ with this meaning subsequently in these notes.

In Section 5A we list the Y_{jm} for $j \le 4$.

2B. Vector spherical harmonics $\vec{Y}_{jlm}(\Omega)$

We define

$$\vec{Y}_{j\ell m}(\Omega) = \sum_{\alpha\beta} \begin{pmatrix} \ell & 1 & j \\ \alpha & \beta & m \end{pmatrix} Y_{\ell\alpha}(\Omega)\, \hat{e}_\beta \tag{4}$$

where $\begin{pmatrix} \ell & 1 & j \\ \alpha & \beta & m \end{pmatrix}$ is a Clebsch-Gordan coefficient (see Rose or Schiff), sometimes written $\langle \ell 1 \alpha\beta \mid \ell 1 j m \rangle$, and the unit vectors $\hat{e}_\beta$ are

$$\hat{e}_1 = \frac{-\hat{x}-i\hat{y}}{\sqrt{2}}, \quad \hat{e}_o = \hat{z}, \quad \hat{e}_{-1} = \frac{\hat{x}-i\hat{y}}{\sqrt{2}} \tag{5}$$

[The $\hat{e}$'s are defined by analogy with the Y_{lm}; compare

$$\sqrt{\frac{4\pi}{3}}\, r\, Y_{11} = -\frac{x+iy}{\sqrt{2}}, \quad \sqrt{\frac{4\pi}{3}}\, r\, Y_{10} = z, \quad \sqrt{\frac{4\pi}{3}}\, r\, Y_{1-1} = \frac{x-iy}{\sqrt{2}}$$]

Note that there are only three possible values for ℓ; ℓ can equal $j-1$, or j, or $j+1$.

Quantum-mechanically, one can think of (4) as a wavefunction for a particle of spin-one, with orbital angular momentum ℓ, total angular momentum j, and z-component of total angular momentum m.

The $\vec{Y}_{j\ell m}(\Omega)$ are orthonormal:

$$\int d\Omega\, \vec{Y}^{*}_{j1m}(\Omega)\cdot\vec{Y}_{j'\ell'm'}(\Omega) = \delta_{jj'}\,\delta_{\ell\ell'}\,\delta_{mm'}$$

The parity of $\vec{Y}_{j\ell m}$ is $(-1)^{\ell+1}$:

$$P\,\vec{Y}_{j\ell m}(\Omega) = -\vec{Y}_{j\ell m}(\pi-\theta,\ \pi+\varphi) = (-1)^{\ell+1}\,\vec{Y}_{j\ell m}(\Omega)$$

Note that we are assuming that the field described by $Y_{j\ell m}$ is a <u>polar</u> vector ($J^P=1^-$ in quantum-mechanical language). If it is an <u>axial</u> vector ($J^P=1^+$), then

$$P\,\vec{Y}_{j\ell m}(\Omega) = +\,\vec{Y}_{j\ell m}(\pi-\theta,\ \pi+\varphi) = (-1)^{\ell}\,\vec{Y}_{j\ell m}(\Omega)$$

and the parity of $\vec{Y}_{j\ell m}$ is $(-1)^{\ell}$.

The vector function $\vec{\Psi}(\vec{x}) = f_\ell(kr)\vec{Y}_{j\ell m}(\Omega)$ obeys the vector Helmholtz equation

$$\left(\nabla^2+k^2\right)\vec{\Psi}\left(\vec{x}\right) = 0 \tag{6}$$

provided $f_\ell(kr)$ is a spherical Bessel function, as in (3).

<u>Other notations</u>:

Jackson: $\vec{X}_{jm}(\Omega) = \vec{Y}_{jjm}(\Omega)$

Rose: $\underset{\sim}{T}_{j\ell m}(\Omega) = \vec{Y}_{j\ell m}(\Omega)$

Morse and Feshbach: $\underset{\sim}{P}_{mj} = N_{jm}\left[\sqrt{\frac{j}{2j+1}}\ \vec{Y}_{jj-1m} - \sqrt{\frac{j+1}{2j+1}}\ \vec{Y}_{jj+1m}\right]$

$$\underset{\sim}{B}_{mj} = N_{jm}\left[\sqrt{\frac{j+1}{2j+1}}\ \vec{Y}_{jj-1m} + \sqrt{\frac{j}{2j+1}}\ \vec{Y}_{jj+1m}\right]$$

$$\underset{\sim}{C}_{mj} = -iN_{jm}\,\vec{Y}_{jjm}$$

where

$$N_{jm} = \sqrt{\frac{4\pi}{2j+1} \frac{(j+|m|)!}{(j-|m|)!}} \cdot \begin{cases} (-1)^m & \text{if } m \geq 0 \\ 1 & \text{if } m < 0 \end{cases}$$

Davis:

$$\vec{R}_{jm} = \sqrt{\frac{j}{2j+1}}\ \vec{Y}_{jj-1m} - \sqrt{\frac{j+1}{2j+1}}\ \vec{Y}_{jj+1m}$$

$$\vec{S}_{jm} = \sqrt{\frac{j+1}{2j+1}}\ \vec{Y}_{jj-1m} + \sqrt{\frac{j}{2j+1}}\ \vec{Y}_{jj+1m}$$

$$\vec{T}_{jm} = \vec{Y}_{jjm}$$

In Section 5B we give the $\vec{Y}_{j\ell m}$ for $j \leq 4$.

2C. Tensor spherical harmonics $\underset{\sim}{T}_{j\ell m}(\Omega)$

Just as the vector spherical harmonics $\vec{Y}_{j\ell m}(\Omega)$ represent the combination of orbital angular momentum (the $Y_{\ell m}(\Omega)$) with spin-one (the unit vectors $\hat{e}_\beta$ of (5)), we define tensor spherical harmonics $\underset{\sim}{T}_{j\ell m}(\Omega)$ by combining orbital angular momentum ℓ with spin-two, i.e., with unit tensors. (When we refer to tensors in these notes, we always mean symmetric, traceless tensors of rank two.)

We thus define

$$\underset{\sim}{T}_{j\ell m}(\Omega) = \sum_{\alpha\beta} \begin{pmatrix} \ell & 2 & j \\ \alpha & \beta & m \end{pmatrix} Y_{\ell\alpha}(\Omega)\, \underset{\sim}{t}_\beta$$

where $\begin{pmatrix} \ell & 2 & j \\ \alpha & \beta & m \end{pmatrix}$ again denotes a Clebsch-Gordan coefficient,

and the unit tensors $\underset{\sim}{t}_\beta$ are:

$$\underset{\sim}{t}_2 = \frac{1}{2}\ (\hat{x}\hat{x} - \hat{y}\hat{y}) + \frac{i}{2}\ (\hat{x}\hat{y} + \hat{y}\hat{x})$$

$$\underset{\sim}{t}_1 = -\frac{1}{2}\ (\hat{x}\hat{z} + \hat{z}\hat{x}) - \frac{i}{2}\ (\hat{y}\hat{z} + \hat{z}\hat{y})$$

$$\underset{\sim}{t}_{o} = \sqrt{\frac{1}{6}}\,(2\,\hat{z}\,\hat{z} - \hat{x}\,\hat{x} - \hat{y}\,\hat{y})$$

$$\underset{\sim}{t}_{-1} = \frac{1}{2}\,(\hat{x}\,\hat{z} + \hat{z}\,\hat{x}) - \frac{i}{2}\,(\hat{y}\,\hat{z} + \hat{z}\,\hat{y})$$

$$\underset{\sim}{t}_{-2} = \frac{1}{2}\,(\hat{x}\,\hat{x} - \hat{y}\,\hat{y}) - \frac{i}{2}\,(\hat{x}\,\hat{y} + \hat{y}\,\hat{x})$$

[The $\underset{\sim}{t}_{\beta}$ are defined by analogy with the $Y_{2m}\,(\Omega)$; note that

$$\sqrt{\frac{8\pi}{15}}\,r^2\,Y_{22}\,(\Omega) = \frac{1}{2}\,(xx - yy) + \frac{i}{2}\,(xy + yx)$$

$$\sqrt{\frac{8\pi}{15}}\,r^2\,Y_{21}\,(\Omega) = -\frac{1}{2}\,(xz + zx) - \frac{i}{2}\,(yz + zy)$$

$$\sqrt{\frac{8\pi}{15}}\,r^2\,Y_{20}\,(\Omega) = \sqrt{\frac{1}{6}}\,\left(2z^2 - x^2 - y^2\right)$$

$$\sqrt{\frac{8\pi}{15}}\,r^2\,Y_{2-1}\,(\Omega) = \frac{1}{2}\,(xz + zx) - \frac{i}{2}\,(yz + zy)$$

$$\sqrt{\frac{8\pi}{15}}\,r^2\,Y_{2-2}\,(\Omega) = \frac{1}{2}\,(xx - yy) - \frac{i}{2}\,(xy + yx)$$]

The $\underset{\sim}{t}_{\beta}$ are normalized so that

$$\underset{\sim}{t}_{\alpha}^{*} : \underset{\sim}{t}_{\beta} = \delta_{\alpha\beta}$$

where the "double dot product" $\underset{\sim}{A}:\underset{\sim}{B}$ of two tensors is defined by

$$\underset{\sim}{A}:\underset{\sim}{B} \equiv \sum_{ij} A_{ij}\,B_{ij}$$

Note that there are now five possible values for ℓ : ℓ can equal $j-2$, or $j-1$, or j, or $j+1$, or $j+2$.

Besides being symmetric and traceless, the $\underset{\sim}{T}_{j\ell m}$ are orthonormal:

$$\int d\Omega\;\underset{\sim}{T}^{*}_{j\ell m}(\Omega) : \underset{\sim}{T}_{j'\ell' m'}(\Omega) = \delta_{jj'}\,\delta_{\ell\ell'}\,\delta_{mm'}$$

The parity of $\underset{\sim}{T}_{j\ell m}$ is $(-1)^{\ell}$: (We assume a 2+ field, not a 2- field!)

$$P\,\underset{\sim}{T}_{j\ell m}(\Omega) = \underset{\sim}{T}_{j\ell m}(\pi-\theta,\ \pi+\varphi) = (-1)^{\ell}\,\underset{\sim}{T}_{j\ell m}(\Omega)$$

The tensor $\underset{\sim}{\Psi}(\vec{x}) = f_{\ell}(kr)\underset{\sim}{T}_{j\ell m}(\Omega)$ obeys the tensor Helmholtz equation

$$\left(\nabla^2 + k^2\right)\,\underset{\sim}{\Psi}\left(\vec{x}\right) = 0$$

provided $f_{\ell}(kr)$ is a spherical Bessel function; see (3).

In Section 5D we list the $\underset{\sim}{T}_{j\ell m}(\Omega)$ for $j \le 4$.

3. Transverse spherical harmonics

3A. Transverse vector spherical harmonics $\vec{Y}^{e}_{jm}(\Omega)$ and $\vec{Y}^{m}_{jm}(\Omega)$

For many purposes, such as the multipole expansion of electromagnetism, it is useful to have vector spherical harmonics with definite (j,m) values which are transverse, i.e., which have no radial component. There are two such vector spherical harmonics:

$$\vec{Y}^{e}_{jm}(\Omega) = \sqrt{\frac{j+1}{2j+1}}\;\vec{Y}_{j\,j-1\,m}(\Omega) + \sqrt{\frac{j}{2j+1}}\;\vec{Y}_{j\,j+1\,m}(\Omega)$$

$$\vec{Y}^{m}_{jm}(\Omega) = \vec{Y}_{jjm}(\Omega)$$

The superscripts e and m stand for "electric" and "magnetic", respectively. In electric (or magnetic) multipole radiation the vector potential $\vec{A}$ is proportional to $\vec{Y}^{e}_{jm}$ (or $\vec{Y}^{m}_{jm}$). These transverse spherical harmonics have opposite parities:

$$P\,\vec{Y}^{e}_{jm} = (-1)^{j}\,\vec{Y}^{e}_{jm}\ ,\quad P\,\vec{Y}^{m}_{jm} = (-1)^{j+1}\,\vec{Y}^{m}_{jm}\quad \text{(for a } 1^{-} \text{ field)}$$

The $\vec{Y}^{e}_{jm}$ and $\vec{Y}^{m}_{jm}$ are orthonormal:

$$\int d\Omega\ \vec{Y}^{e\,*}_{jm}(\Omega)\cdot\vec{Y}^{e}_{j'm'}(\Omega) = \int d\Omega\ \vec{Y}^{m\,*}_{jm}(\Omega)\cdot\vec{Y}^{m}_{j'm'}(\Omega) = \delta_{jj'}\,\delta_{mm'}$$

$$\int d\Omega\ \vec{Y}^{e\,*}_{jm}(\Omega)\cdot\vec{Y}^{m}_{j'm'}(\Omega) = 0$$

Note that

$$\hat{r}\times\vec{Y}^{m}_{jm} = i\,\vec{Y}^{e}_{jm}$$

$$\hat{r}\times\vec{Y}^{e}_{jm} = i\,\vec{Y}^{m}_{jm}$$

Morse and Feshbach's $\underset{\sim}{B}_{mj}$ and $\underset{\sim}{C}_{mj}$ are, to within a normalization constant, our $\vec{Y}^{e}_{jm}$ and $\vec{Y}^{m}_{jm}$, respectively. Also Jackson's $\underset{\sim}{X}_{jm}$ is the same as $\vec{Y}^{m}_{jm}$.

In Section 5C we list the transverse vector spherical harmonics $\vec{Y}^{e}_{jm}(\Omega)$ and $\vec{Y}^{m}_{jm}(\Omega)$ for $j \le 4$.

3B. Transverse tensor spherical harmonics

In describing gravitational multipole radiation, one needs to construct transverse tensor spherical harmonics, i.e., tensor spherical harmonics $\underset{\sim}{T}_{jm}$ such that $\hat{r}.\underset{\sim}{T}_{jm} = 0$. There are again two such harmonics:

$$\underset{\sim}{T}^{e}_{jm}(\Omega) = \sqrt{\frac{(j+1)\ (j+2)}{2(2j-1)(2j+1)}}\ \underset{\sim}{T}_{jj-2m}(\Omega) + \sqrt{\frac{3(j-1)\ (j+2)}{(2j-1)\ (2j+3)}}\ \underset{\sim}{T}_{jjm}(\Omega) + \sqrt{\frac{j(j-1)}{2(2j+1)\ (2j+3)}}\ \underset{\sim}{T}_{jj+2m}(\Omega) \tag{7}$$

$$\underset{\sim}{T}^{m}_{jm}(\Omega) = \sqrt{\frac{j+2}{2j+1}}\ \underset{\sim}{T}_{jj-1m}(\Omega) + \sqrt{\frac{j-1}{2j+1}}\ \underset{\sim}{T}_{jj+1m}(\Omega)$$

[Note that the superscripts e and m on these harmonics have been exchanged from the original convention of J. Soc. Indust. Appl. Math. 10, 768 (1962). Thorne and others have pointed out that the convention (7) is more natural.]

These transverse tensors have opposite parities

$$P\,\underset{\sim}{T}^{e}_{jm} = (-1)^{j}\,\underset{\sim}{T}^{e}_{jm} \quad , \quad P\,\underset{\sim}{T}^{m}_{jm} = (-1)^{j+1}\,\underset{\sim}{T}^{m}_{jm} \quad \text{(for a } 2^{+} \text{ field)}$$

and are orthonormal

$$\int d\Omega\, \underset{\sim}{T}^{e\,*}_{jm}(\Omega) : \underset{\sim}{T}^{e}_{j'm'}(\Omega) = \int d\Omega\, \underset{\sim}{T}^{m\,*}_{jm}(\Omega) : \underset{\sim}{T}^{m}_{j'm'}(\Omega) = \delta_{jj'}\,\delta_{mm'}$$

$$\int d\Omega\, \underset{\sim}{T}^{e\,*}_{jm}(\Omega) : \underset{\sim}{T}^{m}_{j'm'}(\Omega) = 0$$

In Section 5E we list the $\underset{\sim}{T}^{e}_{jm}(\Omega)$ and $\underset{\sim}{T}^{m}_{jm}(\Omega)$ for $j \le 4$.

4. Properties

4A. Recursion relations

$$Y_{j,m+1}(\Omega) = \frac{e^{i\varphi}}{\sqrt{(j-m)(j+m+1)}} \left[\frac{\partial}{\partial\theta} - m\cot\theta\right] Y_{jm}(\Omega)$$

$$Y_{j,m-1}(\Omega) = \frac{e^{-i\varphi}}{\sqrt{(j+m)(j-m+1)}} \left[-\frac{\partial}{\partial\theta} - m\cot\theta\right] Y_{jm}(\Omega)$$

$$Y_{j+1,m}(\Omega) = \sqrt{\frac{2j+3}{(2j+1)(j+m+1)(j-m+1)}}\; \sin\theta \left[\frac{\partial}{\partial\theta} + (j+1)\cot\theta\right] Y_{jm}(\Omega)$$

$$Y_{j-1,m}(\Omega) = \sqrt{\frac{2j-1}{(2j+1)(j+m)(j-m)}}\; \sin\theta \left[-\frac{\partial}{\partial\theta} + j\cot\theta\right] Y_{jm}(\Omega)$$

$$Y_{j+1,m+1}(\Omega) = e^{i\varphi}\sqrt{\frac{2j+3}{(2j+1)(j+m+1)(j+m+2)}} \left[\cos\theta\,\frac{\partial}{\partial\theta} - (j+1)\sin\theta - \frac{m}{\sin\theta}\right] Y_{jm}(\Omega)$$

$$Y_{j+1,m-1}(\Omega) = e^{-i\varphi} \sqrt{\frac{2j+3}{(2j+1)(j-m+1)(j-m+2)}} \left[-\cos\theta \frac{\partial}{\partial\theta} + (j+1)\sin\theta - \frac{m}{\sin\theta}\right] Y_{jm}(\Omega)$$

$$Y_{j-1,m+1}(\Omega) = e^{i\varphi} \sqrt{\frac{2j-1}{(2j+1)(j-m-1)(j-m)}} \left[\cos\theta \frac{\partial}{\partial\theta} + j\sin\theta - \frac{m}{\sin\theta}\right] Y_{jm}(\Omega)$$

$$Y_{j-1,m-1}(\Omega) = e^{-i\varphi} \sqrt{\frac{2j-1}{(2j+1)(j+m)(j+m-1)}} \left[-\cos\theta \frac{\partial}{\partial\theta} - j\sin\theta - \frac{m}{\sin\theta}\right] Y_{jm}(\Omega)$$

4B. Properties involving the unit vector $\hat{r}$

$$\hat{r}\, Y_{jm} = \sqrt{\frac{j}{2j+1}}\, \vec{Y}_{jj-1m} - \sqrt{\frac{j+1}{2j+1}}\, \vec{Y}_{jj+1m}$$

$$\hat{r}\cdot\vec{Y}_{j\ell m} = \begin{cases} \sqrt{\frac{j}{2j+1}}\, Y_{jm} & (\ell = j-1) \\ 0 & (\ell = j) \\ -\sqrt{\frac{j+1}{2j+1}}\, Y_{jm} & (\ell = j+1) \end{cases}$$

$$\hat{r}\times\vec{Y}_{j\ell m} = \begin{cases} i\sqrt{\frac{j+1}{2j+1}}\, \vec{Y}_{jjm} & (\ell = j-1) \\ i\left[\sqrt{\frac{j+1}{2j+1}}\, \vec{Y}_{jj-1m} + \sqrt{\frac{j}{2j+1}}\, \vec{Y}_{jj+1m}\right] & (\ell = j) \\ i\sqrt{\frac{j}{2j+1}}\, \vec{Y}_{jjm} & (\ell = j+1) \end{cases}$$

$$\hat{r}\cdot\underset{\sim}{T}_{j\ell m}=\begin{cases}\sqrt{\dfrac{j-1}{2j-1}}\;\vec{Y}_{jj-1m} & (\ell=j-2)\\[2ex] \sqrt{\dfrac{j-1}{2(2j+1)}}\;\vec{Y}_{jjm} & (\ell=j-1)\\[2ex] -\sqrt{\dfrac{(j+1)(2j+3)}{6(2j+1)(2j-1)}}\;\vec{Y}_{jj-1m}+\sqrt{\dfrac{j(2j-1)}{6(2j+1)(2j+3)}}\;\vec{Y}_{jj+1m} & (\ell=j)\\[2ex] -\sqrt{\dfrac{j+2}{2(2j+1)}}\;\vec{Y}_{jjm} & (\ell=j+1)\\[2ex] -\sqrt{\dfrac{j+2}{2j+3}}\;\vec{Y}_{jj+1m} & (\ell=j+2)\end{cases}$$

4C. Products

(scalar-scalar) $$Y_{\ell m}\,Y_{\ell' m'}=\sum_{\lambda}\sqrt{\frac{(2\ell+1)(2\ell'+1)}{4\pi(2\lambda+1)}}\begin{pmatrix}\ell & \ell' & \lambda\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell & \ell' & \lambda\\ m & m' & m+m'\end{pmatrix}Y_{\lambda\, m+m'}$$

(scalar-vector) $$Y_{\ell m}\,\vec{Y}_{j'\ell' m'}=\sum_{\lambda f}\sqrt{\frac{(2\ell+1)(2\ell'+1)(2j'+1)}{4\pi}}\begin{pmatrix}\ell & \ell' & \lambda\\ 0 & 0 & 0\end{pmatrix}$$

$$W(\ell\,\ell' f 1;\lambda\, j')\begin{pmatrix}\ell & j' & f\\ m & m' & m+m'\end{pmatrix}\vec{Y}_{f\lambda\, m+m'}$$

where $W(\ell\,\ell' f1;\lambda\, j')$ is a Racah coefficient (see Rose).

(vector-vector) $$\vec{Y}_{j\ell m}\cdot\vec{Y}_{j'\ell' m'}=-\sum_{\lambda}\sqrt{\frac{(2\ell+1)(2j+1)(2\ell'+1)(2j'+1)}{4\pi(2\lambda+1)}}\begin{pmatrix}\ell & \ell' & \lambda\\ 0 & 0 & 0\end{pmatrix}$$

$$W(\ell 1\lambda j';j\,\ell')\begin{pmatrix}j & j' & \lambda\\ m & m' & m+m'\end{pmatrix}Y_{\lambda,\, m+m'}$$

$$\vec{Y}_{j\ell m}\times\vec{Y}_{j'\ell' m'}=i\sqrt{\frac{3}{2\pi}}\sqrt{(2\ell+1)(2j+1)(2\ell'+1)(2j'+1)}\sum_{JL}\begin{pmatrix}\ell & \ell' & L\\ 0 & 0 & 0\end{pmatrix}$$

$$X \begin{pmatrix} \ell & 1 & j \\ \ell' & 1 & j' \\ L & 1 & J \end{pmatrix} \begin{pmatrix} j & j' & J \\ m & m' & m+m' \end{pmatrix} \vec{Y}_{JL\,m+m'}$$

where the symbol $X \begin{pmatrix} \ell & 1 & j \\ \ell' & 1 & j' \\ L & 1 & J \end{pmatrix}$ is an X-coefficient, or a 9-j coefficient (see Rose).

(vector-tensor) $\vec{Y}_{j\ell m} \cdot \underset{\sim}{T}_{j'\ell' m'} = -\sqrt{\frac{5}{4\pi}} \sqrt{(2\ell+1)(2j+1)(2\ell'+1)(2j'+1)}$

$$\sum_{JL} \begin{pmatrix} \ell & \ell' & L \\ 0 & 0 & 0 \end{pmatrix} X \begin{pmatrix} \ell & 1 & j \\ \ell' & 2 & j' \\ L & 1 & J \end{pmatrix} \begin{pmatrix} j & j' & J \\ m & m' & m+m' \end{pmatrix} \vec{Y}_{J\,L\,m+m'}$$

4D. Differential properties

<u>gradient of a scalar</u> $\vec{\nabla} f(r) Y_{jm}(\Omega) = \sqrt{\frac{j}{2j+1}} \left[f'(r) + \frac{j+1}{r} f(r) \right] \vec{Y}_{jj-1m}$

$$- \sqrt{\frac{j+1}{2j+1}} \left[f'(r) - \frac{j}{r} f(r) \right] \vec{Y}_{jj+1m} \qquad (7)$$

If $f(r)$ equals a spherical Bessel function $f_j(kr)$ (see (3)), the identities

$$\frac{d}{dx} f_j(x) + \frac{j+1}{x} f_j(x) = f_{j-1}(x)$$

$$\frac{d}{dx} f_j(x) - \frac{j}{x} f_j(x) = -f_{j+1}(x)$$

enable (7) to be written

$$\vec{\nabla} f_j(kr) Y_{jm}(\Omega) = k \left[\sqrt{\frac{j}{2j+1}} f_{j-1}(kr) \vec{Y}_{jj-1m} + \sqrt{\frac{j+1}{2j+1}} f_{j+1}(kr) \vec{Y}_{jj+1m} \right]$$

Note how the subscript of the $f_j(kr)$ has "conspired" to remain equal to the ℓ of the vector spherical harmonic it multiplies. Thus if $f(r)Y_{jm}(\Omega)$

obeys the scalar Helmholtz equation (2), then $\vec{\nabla} f(r)Y_{jm}(\Omega)$ obeys the vector Helmholtz equation (6), as of course it must.

divergence of a vector

$$\vec{\nabla} \cdot f(r)\vec{Y}_{j\ell m}(\Omega) = \begin{cases} \sqrt{\dfrac{j}{2j+1}}\left[f'(r) - \dfrac{j-1}{r} f(r)\right] Y_{jm} & (\ell = j-1) \\ 0 & (\ell = j) \\ -\sqrt{\dfrac{j+1}{2j+1}}\left[f'(r) + \dfrac{j+2}{r} f(r)\right] Y_{jm} & (\ell = j+1) \end{cases}$$

If f(r) equals a spherical Bessel function $f_\ell(kr)$, these become

$$\vec{\nabla} \cdot f_\ell(kr)\, \vec{Y}_{j\ell m}(\Omega) = \begin{cases} -k\sqrt{\dfrac{j}{2j+1}}\, f_j(kr)\, Y_{jm} & (\ell = j-1) \\ 0 & (\ell = j) \\ -k\sqrt{\dfrac{j+1}{2j+1}}\, f_j(kr)\, Y_{jm} & (\ell = j+1) \end{cases}$$

curl of a vector

$$\vec{\nabla} \times f(r)\,\vec{Y}_{jlm}(r) = \begin{cases} i\sqrt{\dfrac{j+1}{2j+1}}\left[f'(r) - \dfrac{j-1}{r} f(r)\right] \vec{Y}_{jjm} & (\ell = j-1) \\ i\left\{\sqrt{\dfrac{j+1}{2j+1}}\left[f'(r) + \dfrac{j+1}{r} f(r)\right] \vec{Y}_{jj-1m} + \sqrt{\dfrac{j}{2j+1}}\left[f'(r) - \dfrac{j}{r} f(r)\right] \vec{Y}_{jj+1m}\right\} & (\ell = j) \\ i\sqrt{\dfrac{j}{2j+1}}\left[f'(r) + \dfrac{j+2}{r} f(r)\right] \vec{Y}_{jjm} & (\ell = j+1) \end{cases}$$

If f(r) equals a spherical Bessel function $f_\ell(kr)$, these become

$$\vec{\nabla} \times f_\ell(kr)\, \vec{Y}_{j\ell m}(r) = \begin{cases} -ik\sqrt{\frac{j+1}{2j+1}}\; f_j(kr)\, \vec{Y}_{jjm} & (\ell = j-1) \\ +ik\left\{\sqrt{\frac{j+1}{2j+1}}\; f_{j-1}(kr)\, \vec{Y}_{jj-1m} - \sqrt{\frac{j}{2j+1}}\; f_{j+1}(kr)\, \vec{Y}_{jj+1m}\right\} & (\ell = j) \\ ik\sqrt{\frac{j}{2j+1}}\; f_j(kr)\, \vec{Y}_{jjm} & (\ell = j+1) \end{cases}$$

By combining the previous formulas, we can obtain as checks:

<u>Laplacian of a scalar</u> $(\nabla^2\Psi = \vec{\nabla} \cdot \vec{\nabla}\Psi)$

$$\nabla^2 f(r)\, Y_{jm}(\Omega) = \left[f''(r) + \frac{2}{r} f'(r) - \frac{j(j+1)}{r^2} f(r) \right] Y_{jm}$$

If f(r) is a spherical Bessel function $f_j(kr)$, we have of course

$$\nabla^2 f_j(kr)\, Y_{jm}(\Omega) = -k^2 f_j(kr)\, Y_{jm}$$

<u>Laplacian of a vector</u> $(\nabla^2 \vec{\Psi} = \vec{\nabla}\, \vec{\nabla} \cdot \vec{\Psi} - \vec{\nabla} \times (\vec{\nabla} \times \vec{\Psi}))$

$$\nabla^2 f(r)\, \vec{Y}_{j\ell m}(\Omega) = \left[f''(r) + \frac{2}{r} f'(r) - \frac{\ell(\ell+1)}{r^2} f(r) \right] \vec{Y}_{j\ell m}$$

If f(r) is a spherical Bessel function $f_\ell(kr)$, then of course

$$\nabla^2 f_\ell(kr)\, \vec{Y}_{j\ell m}(\Omega) = -k^2 f_\ell(kr)\, \vec{Y}_{j\ell m}$$

[end of checks]

<u>gradient of a vector</u> (equals a tensor!)

Since our tensor spherical harmonics are symmetric and traceless, we define the tensor $(\vec{\nabla}\vec{f})$ as

$$(\vec{\nabla}\vec{f})_{ij} = \frac{1}{2}(\nabla_i f_j + \nabla_j f_i) - \frac{1}{3}\delta_{ij}\, \vec{\nabla} \cdot \vec{f}$$

Then

$$\left(\vec{\nabla} f(r) Y_{j\ell m}(\Omega)\right) = \begin{cases} \sqrt{\frac{j-1}{2j-1}}\left[f'(r)+\frac{j}{r}f(r)\right]\underset{\sim}{T}_{jj-2m} - \sqrt{\frac{(j+1\,(2j+3)}{6(2j+1)\,(2j-1)}}\left[f'(r)-\frac{j-1}{r}f(r)\right]\underset{\sim}{T}_{jjm} & (\ell=j-1) \\ \sqrt{\frac{j-1}{2(2j+1)}}\left[f'(r)+\frac{j+1}{r}f(r)\right]\underset{\sim}{T}_{jj-1m} - \sqrt{\frac{j+2}{2(2j+1)}}\left[f'(r)-\frac{j}{r}f(r)\right]\underset{\sim}{T}_{jj+1m} & (\ell=j) \\ \sqrt{\frac{j(2j-1)}{6(2j+1)\,(2j+3)}}\left[f'(r)+\frac{j+2}{r}f(r)\right]\underset{\sim}{T}_{jjm} - \sqrt{\frac{j+2}{2j+3}}\left[f'(r)-\frac{j+1}{r}f(r)\right]\underset{\sim}{T}_{jj+2m} & (\ell=j+1) \end{cases}$$

If f(r) equals a spherical Bessel function $f_\ell(kr)$, these become

$$\vec{\nabla} f_\ell(kr)\,\vec{Y}_{j\ell m}(\Omega) = \begin{cases} k\left[\sqrt{\frac{j-1}{2j-1}}\,f_{j-2}(kr)\,\underset{\sim}{T}_{jj-2m} + \sqrt{\frac{(j+1)\,(2j+3)}{6(2j+1)\,(2j-1)}}\,f_j(kr)\,\underset{\sim}{T}_{jjm}\right] & (\ell=j-1) \\ k\left[\sqrt{\frac{j-1}{2(2j+1)}}\,f_{j-1}(kr)\,\underset{\sim}{T}_{jj-1m} + \sqrt{\frac{j+2}{2(2j+1)}}\,f_{j+1}(kr)\,\underset{\sim}{T}_{jj+1m}\right] & (\ell=j) \\ k\left[\sqrt{\frac{j(2j-1)}{6(2j+1)\,(2j+3)}}\,f_j(kr)\,\underset{\sim}{T}_{jjm} + \sqrt{\frac{j+2}{2j+3}}\,f_{j+2}(kr)\,\underset{\sim}{T}_{jj+2m}\right] & (\ell=j+1) \end{cases}$$

<u>divergence of a tensor</u>

$$\vec{\nabla}\cdot f(r)\underset{\sim}{T}_{j\ell m}(\Omega) = \begin{cases} \sqrt{\frac{j-1}{2j-1}}\left[f'(r)-\frac{j-2}{r}f(r)\right]\vec{Y}_{jj-1m} & (\ell=j-2) \\ \sqrt{\frac{j-1}{2(2j+1)}}\left[f'(r)-\frac{j-1}{r}f(r)\right]\vec{Y}_{jjm} & (\ell=j-1) \\ -\sqrt{\frac{(j+1)\,(2j+3)}{6(2j+1)\,(2j-1)}}\left[f'(r)+\frac{j+1}{r}f(r)\right]\vec{Y}_{jj-1m} + \sqrt{\frac{j(2j-1)}{6(2j+1)\,(2j+3)}}\left[f'(r)-\frac{j}{r}f(r)\right]\vec{Y}_{jj+1m} & (\ell=j) \\ -\sqrt{\frac{j+2}{2(2j+1)}}\left[f'(r)+\frac{j+2}{r}f(r)\right]\vec{Y}_{jjm} & (\ell=j+1) \\ -\sqrt{\frac{j+2}{2j+3}}\left[f'(r)+\frac{j+3}{r}f(r)\right]\vec{Y}_{jj+1m} & (\ell=j+2) \end{cases}$$

If f(r) equals a spherical Bessel function $f_\ell(kr)$, these become

$$\vec{\nabla} f_\ell(kr)\, \underset{\sim}{T}_{j\ell m}(\Omega) = \begin{cases} -k\sqrt{\dfrac{j-1}{2j-1}}\; f_{j-1}(kr)\, \vec{Y}_{jj-1m} & (\ell = j-2) \\[2ex] -k\sqrt{\dfrac{j-1}{2(2j+1)}}\; f_j(kr)\, \vec{Y}_{jjm} & (\ell = j-1) \\[2ex] -k\left[\sqrt{\dfrac{(j+1)(2j+3)}{6(2j+1)(2j-1)}}\; f_{j-1}(kr)\, \vec{Y}_{jj-1m} + \sqrt{\dfrac{j(2j-1)}{6(2j+1)(2j+3)}}\; f_{j+1}(kr)\vec{Y}_{jj+1m}\right] & (\ell = j) \\[2ex] -k\sqrt{\dfrac{j+2}{2(2j+1)}}\; f_j(kr)\, \vec{Y}_{jjm} & (\ell = j+1) \\[2ex] -k\sqrt{\dfrac{j+2}{2j+3}}\; f_{j+1}(kr)\, \vec{Y}_{jj+1m} & (\ell = j+2) \end{cases}$$

5. Tables

5A. $Y_{jm}(\Omega)$

Note the symmetry $Y_{j-m} = (-1)^m Y^*_{jm}$

j	m	Y_{jm}
0	0	$\sqrt{\frac{1}{4\pi}}$
1	1	$-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi}$
1	0	$\sqrt{\frac{3}{4\pi}}\cos\theta$
1	-1	$\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi}$
2	2	$\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\varphi}$
2	1	$-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\varphi}$
2	0	$\sqrt{\frac{5}{16\pi}}(3\cos^2\theta-1)$
2	-1	$\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\varphi}$
2	-2	$\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{-2i\varphi}$
3	3	$-\sqrt{\frac{35}{64\pi}}\sin^3\theta\, e^{3i\varphi}$
3	2	$\sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta\, e^{2i\varphi}$
3	1	$-\sqrt{\frac{21}{64\pi}}(5\cos^2\theta-1)\sin\theta\, e^{i\varphi}$
3	0	$\sqrt{\frac{7}{16\pi}}(5\cos^2\theta-3)\cos\theta$
3	-1	$\sqrt{\frac{21}{64\pi}}(5\cos^2\theta-1)\sin\theta\, e^{-i\varphi}$
3	-2	$\sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\, e^{-2i\varphi}$

j	m	Y_{jm}
3	-3	$\sqrt{\frac{35}{64\pi}}\sin^3\theta\, e^{-3i\varphi}$
4	4	$\sqrt{\frac{315}{512\pi}}\sin^4\theta\, e^{4i\varphi}$
4	3	$-\sqrt{\frac{315}{64\pi}}\sin^3\theta\cos\theta\, e^{3i\varphi}$
4	2	$\sqrt{\frac{45}{128\pi}}(7\cos^2\theta-1)\sin^2\theta\, e^{2i\varphi}$
4	1	$-\sqrt{\frac{45}{64\pi}}(7\cos^2\theta-3)\sin\theta\cos\theta\, e^{i\varphi}$
4	0	$\sqrt{\frac{9}{256\pi}}(35\cos^4\theta-30\cos^2\theta+3)$
4	-1	$\sqrt{\frac{45}{64\pi}}(7\cos^2\theta-3)\sin\theta\cos\theta\, e^{-i\varphi}$
4	-2	$\sqrt{\frac{45}{128\pi}}(7\cos^2\theta-1)\sin^2\theta\, e^{-2i\varphi}$
4	-3	$\sqrt{\frac{315}{64\pi}}\sin^3\theta\cos\theta\, e^{-3i\varphi}$
4	-4	$\sqrt{\frac{315}{512\pi}}\sin^4\theta\, e^{-4i\varphi}$

5B. $\vec{Y}_{j\ell m}$

For each $(j\ell m)$ we give an overall factor and then the $\hat{r}$, $\hat{\theta}$, and $\hat{\varphi}$ components of the vector harmonic. At the end of this section we give explicit expressions for the components. Note the symmetry

$\vec{Y}_{j\ell -m} = (-)^{j+\ell+m+1}\,\vec{Y}^{\,*}_{j\ell m}$

j	ℓ	m	factor	$\hat{r}$	$\hat{\theta}$	$\hat{\varphi}$
0	1	0	$\sqrt{\frac{1}{4\pi}}$	-1	0	0
1	0	1	$\sqrt{\frac{1}{8\pi}}\, e^{i\varphi}$	$-\sin\theta$	$-\cos\theta$	$-i$
1	0	0	$\sqrt{\frac{1}{4\pi}}$	$\cos\theta$	$-\sin\theta$	0
1	0	-1	$\sqrt{\frac{1}{8\pi}}\, e^{-i\varphi}$	$\sin\theta$	$\cos\theta$	$-i$
1	1	1	$\sqrt{\frac{3}{16\pi}}\, e^{i\varphi}$	0	1	$i\cos\theta$
1	1	0	$\sqrt{\frac{3}{8\pi}}\, \sin\theta$	0	0	i
1	1	-1	$\sqrt{\frac{3}{16\pi}}\, e^{-i\varphi}$	0	1	$-i\cos\theta$
1	2	1	$\sqrt{\frac{1}{16\pi}}\, e^{i\varphi}$	$2\sin\theta$	$-\cos\theta$	$-i$
1	2	0	$\sqrt{\frac{1}{8\pi}}$	$-2\cos\theta$	$-\sin\theta$	0
1	2	-1	$\sqrt{\frac{1}{16\pi}}\, e^{-i\varphi}$	$-2\sin\theta$	$\cos\theta$	$-i$
2	1	2	$\sqrt{\frac{3}{16\pi}}\, \sin\theta\, e^{2i\varphi}$	$\sin\theta$	$\cos\theta$	i
2	1	1	$\sqrt{\frac{3}{16\pi}}\, e^{i\varphi}$	$-2\sin\theta\cos\theta$	$1-2\cos^2\theta$	$-i\cos\theta$
2	1	0	$\sqrt{\frac{1}{8\pi}}$	$3\cos^2\theta-1$	$-3\sin\theta\cos\theta$	0

j	ℓ	m	factor	$\hat{r}$	$\hat{\theta}$	$\hat{\varphi}$
2	1	-1	$\sqrt{\frac{3}{16\pi}}\, e^{-i\varphi}$	$2 \sin\theta \cos\theta$	$2 \cos^2\theta-1$	$-i \cos\theta$
2	1	-2	$\sqrt{\frac{3}{16\pi}} \sin\theta\, e^{-2i\varphi}$	$\sin\theta$	$\cos\theta$	$-i$
2	2	2	$\sqrt{\frac{5}{16\pi}} \sin\theta\, e^{2i\varphi}$	0	-1	$-i \cos\theta$
2	2	1	$\sqrt{\frac{5}{16\pi}}\, e^{i\varphi}$	0	$\cos\theta$	$i(2\cos^2\theta-1)$
2	2	0	$\sqrt{\frac{15}{8\pi}} \sin\theta$	0	0	$i \cos\theta$
2	2	-1	$\sqrt{\frac{5}{16\pi}}\, e^{-i\varphi}$	0	$\cos\theta$	$i(1-2\cos^2\theta)$
2	2	-2	$\sqrt{\frac{5}{16\pi}} \sin\theta\, e^{-2i\varphi}$	0	1	$-i \cos\theta$
2	3	2	$\sqrt{\frac{1}{32\pi}} \sin\theta\, e^{2i\varphi}$	$-3 \sin\theta$	$2 \cos\theta$	$2i$
2	3	1	$\sqrt{\frac{1}{8\pi}}\, e^{i\varphi}$	$3 \sin\theta \cos\theta$	$1-2\cos^2\theta$	$-i \cos\theta$
2	3	0	$\sqrt{\frac{3}{16\pi}}$	$1-3\cos^2\theta$	$-2 \sin\theta \cos\theta$	0
2	3	-1	$\sqrt{\frac{1}{8\pi}}\, e^{-i\varphi}$	$-3 \sin\theta \cos\theta$	$2\cos^2\theta-1$	$-i \cos\theta$
2	3	-2	$\sqrt{\frac{1}{32\pi}} \sin\theta\, e^{-2i\varphi}$	$-3 \sin\theta$	$2 \cos\theta$	$-2i$
3	2	3	$\sqrt{\frac{15}{64\pi}} \sin^2\theta\, e^{3i\varphi}$	$-\sin\theta$	$-\cos\theta$	$-i$
3	2	2	$\sqrt{\frac{5}{32\pi}} \sin\theta\, e^{2i\varphi}$	$3 \sin\theta \cos\theta$	$3\cos^2\theta-1$	$2i \cos\theta$
3	2	1	$\sqrt{\frac{1}{64\pi}}\, e^{i\varphi}$	$\sin\theta(3-15\cos^2\theta)$	$\cos\theta\,(11-15\cos^2\theta)$	$i(1-5\cos^2\theta)$
3	2	0	$\sqrt{\frac{3}{16\pi}}$	$\cos\theta(5\cos^2\theta-3)$	$-\sin\theta(5\cos^2\theta-1)$	0
3	2	-1	$\sqrt{\frac{1}{64\pi}}\, e^{-i\varphi}$	$\sin\theta(15\cos^2\theta-3)$	$\cos\theta(15\cos^2\theta-11)$	$i(1-5\cos^2\theta)$
3	2	-2	$\sqrt{\frac{5}{32\pi}} \sin\theta\, e^{-2i\varphi}$	$3 \sin\theta \cos\theta$	$3\cos^2\theta-1$	$-2i \cos\theta$

j	ℓ	m	factor	$\hat{r}$	$\hat{\theta}$	$\hat{\varphi}$
3	2	-3	$\sqrt{\frac{15}{64\pi}}\ \sin^2\theta\ e^{-3i\varphi}$	$\sin\theta$	$\cos\theta$	$-i$
3	3	3	$\sqrt{\frac{105}{256\pi}}\ \sin^2\theta\ e^{3i\varphi}$	0	1	$i\cos\theta$
3	3	2	$\sqrt{\frac{35}{128\pi}}\ \sin\theta\ e^{2i\varphi}$	0	$-2\cos\theta$	$i(1-3\cos^2\theta)$
3	3	1	$\sqrt{\frac{7}{256\pi}}\ e^{i\varphi}$	0	$5\cos^2\theta-1$	$i\cos\theta(15\cos^2\theta-11)$
3	3	0	$\sqrt{\frac{21}{64\pi}}\ \sin\theta$	0	0	$i(5\cos^2\theta-1)$
3	3	-1	$\sqrt{\frac{7}{256\pi}}\ e^{-i\varphi}$	0	$5\cos^2\theta-1$	$i\cos\theta(11-15\cos^2\theta)$
3	3	-2	$\sqrt{\frac{35}{128\pi}}\ \sin\theta\ e^{-2i\varphi}$	0	$2\cos\theta$	$i(1-3\cos^2\theta)$
3	3	-3	$\sqrt{\frac{105}{256\pi}}\ \sin^2\theta\ e^{-3i\varphi}$	0	1	$-i\cos\theta$
3	4	3	$\sqrt{\frac{5}{256\pi}}\ \sin^2\theta\ e^{3i\varphi}$	$4\sin\theta$	$-3\cos\theta$	$-3i$
3	4	2	$\sqrt{\frac{15}{128\pi}}\ \sin\theta\ e^{2i\varphi}$	$-4\sin\theta\cos\theta$	$3\cos^2\theta-1$	$2i\cos\theta$
3	4	1	$\sqrt{\frac{3}{256\pi}}\ e^{i\varphi}$	$\sin\theta\ (20\cos^2\theta-4)$	$\cos\theta\ (11-15\cos^2\theta)$	$i(1-5\cos^2\theta)$
3	4	0	$\sqrt{\frac{1}{64\pi}}$	$\cos\theta(12-20\cos^2\theta)$	$\sin\theta(3-15\cos^2\theta)$	0
3	4	-1	$\sqrt{\frac{3}{256\pi}}\ e^{-i\varphi}$	$\sin\theta(4-20\cos^2\theta)$	$\cos\theta(15\cos^2\theta-11)$	$i(1-5\cos^2\theta)$
3	4	-2	$\sqrt{\frac{15}{128\pi}}\ e^{-2i\varphi}\sin\theta$	$-4\sin\theta\cos\theta$	$3\cos^2\theta-1$	$-2i\cos\theta$
3	4	-3	$\sqrt{\frac{5}{256\pi}}\ \sin^2\theta\ e^{-3i\varphi}$	$-4\sin\theta$	$3\cos\theta$	$-3i$
4	3	4	$\sqrt{\frac{35}{128\pi}}\ \sin^3\theta\ e^{4i\varphi}$	$\sin\theta$	$\cos\theta$	i
4	3	3	$\sqrt{\frac{35}{256\pi}}\ \sin^2\theta\ e^{3i\varphi}$	$-4\sin\theta\cos\theta$	$1-4\cos^2\theta$	$-3i\cos\theta$

j	ℓ	m	factor	$\hat{r}$	$\hat{\theta}$	$\hat{\varphi}$
4	3	2	$\sqrt{\frac{5}{128\pi}}\ \sin\theta\, e^{2i\varphi}$	$2\sin\theta(7\cos^2\theta-1)$	$2\cos\theta(7\cos^2\theta-4)$	$i(7\cos^2\theta-1)$
4	3	1	$\sqrt{\frac{5}{256\pi}}\ e^{i\varphi}$	$4\sin\theta\cos\theta(3-7\cos^2\theta)$	$-3+27\cos^2\theta-28\cos^4\theta$	$i\cos\theta(3-7\cos^2\theta)$
4	3	0	$\sqrt{\frac{1}{64\pi}}$	$35\cos^4\theta-30\cos^2\theta+3$	$5\sin\theta\cos\theta(3-7\cos^2\theta)$	0
4	3	-1	$\sqrt{\frac{5}{256\pi}}\ e^{-i\varphi}$	$4\sin\theta\cos\theta(7\cos^2\theta-3)$	$3-27\cos^2\theta+28\cos^4\theta$	$i\cos\theta(3-7\cos^2\theta)$
4	3	-2	$\sqrt{\frac{5}{128\pi}}\ \sin\theta\, e^{-2i\varphi}$	$2\sin\theta(7\cos^2\theta-1)$	$2\cos\theta(7\cos^2\theta-4)$	$i(1-7\cos^2\theta)$
4	3	-3	$\sqrt{\frac{35}{256\pi}}\ \sin^2\theta\, e^{-3i\varphi}$	$4\sin\theta\cos\theta$	$4\cos^2\theta-1$	$-3i\cos\theta$
4	3	-4	$\sqrt{\frac{35}{128\pi}}\ \sin^3\theta\, e^{-4i\varphi}$	$\sin\theta$	$\cos\theta$	$-i$
4	4	4	$\sqrt{\frac{63}{128\pi}}\ \sin^3\theta\, e^{4i\varphi}$	0	-1	$-i\cos\theta$
4	4	3	$\sqrt{\frac{63}{256\pi}}\ \sin^2\theta\, e^{3i\varphi}$	0	$3\cos\theta$	$i(4\cos^2\theta-1)$
4	4	2	$\sqrt{\frac{9}{128\pi}}\ \sin\theta\, e^{2i\varphi}$	0	$1-7\cos^2\theta$	$-2i\cos\theta(7\cos^2\theta-4)$
4	4	1	$\sqrt{\frac{9}{256\pi}}\ e^{i\varphi}$	0	$\cos\theta(7\cos^2\theta-3)$	$i(3-27\cos^2\theta+28\cos^4\theta)$
4	4	0	$\sqrt{\frac{45}{64\pi}}\ \sin\theta$	0	0	$i\cos\theta(7\cos^2\theta-3)$
4	4	-1	$\sqrt{\frac{9}{256\pi}}\ e^{-i\varphi}$	0	$\cos\theta\,(7\cos^2\theta-3)$	$i(-3+27\cos^2\theta-28\cos^4\theta)$
4	4	-2	$\sqrt{\frac{9}{128\pi}}\ \sin\theta\, e^{-2i\varphi}$	0	$7\cos^2\theta-1$	$-2i\cos\theta(7\cos^2\theta-4)$
4	4	-3	$\sqrt{\frac{63}{256\pi}}\ \sin^2\theta\, e^{-3i\varphi}$	0	$3\cos\theta$	$i(1-4\cos^2\theta)$
4	4	-4	$\sqrt{\frac{63}{128\pi}}\ \sin^3\theta\, e^{-4i\varphi}$	0	1	$-i\cos\theta$
4	5	4	$\sqrt{\frac{7}{512\pi}}\ \sin^3\theta\, e^{4i\varphi}$	$-5\sin\theta$	$4\cos\theta$	$4i$
4	5	3	$\sqrt{\frac{7}{64\pi}}\ \sin^2\theta\, e^{3i\varphi}$	$5\sin\theta\cos\theta$	$1-4\cos^2\theta$	$-3i\cos\theta$

j	ℓ	m	factor	$\hat{r}$	$\hat{\theta}$	$\hat{\varphi}$
4	5	2	$\sqrt{\frac{1}{128\pi}}\sin\theta\, e^{2i\varphi}$	$5\sin\theta(1-7\cos^2\theta)$	$4\cos\theta(7\cos^2\theta-4)$	$2i(7\cos^2\theta-1)$
4	5	1	$\sqrt{\frac{1}{64\pi}}\, e^{i\varphi}$	$5\sin\theta\cos\theta(7\cos^2\theta-3)$	$-3+27\cos^2\theta-28\cos^4\theta$	$i\cos\theta(3-7\cos^2\theta)$
4	5	0	$\sqrt{\frac{5}{256\pi}}$	$-3+30\cos^2\theta-35\cos^4\theta$	$4\sin\theta\cos\theta(3-7\cos^2\theta)$	0
4	5	-1	$\sqrt{\frac{1}{64\pi}}\, e^{-i\varphi}$	$5\sin\theta\cos\theta(3-7\cos^2\theta)$	$3-27\cos^2\theta+28\cos^4\theta$	$i\cos\theta(3-7\cos^2\theta)$
4	5	-2	$\sqrt{\frac{1}{128\pi}}\sin\theta\, e^{-2i\varphi}$	$5\sin\theta(1-7\cos^2\theta)$	$4\cos\theta(7\cos^2\theta-4)$	$2i(1-7\cos^2\theta)$
4	5	-3	$\sqrt{\frac{7}{64\pi}}\sin^2\theta\, e^{-3i\varphi}$	$-5\sin\theta\cos\theta$	$4\cos^2\theta-1$	$-3i\cos\theta$
4	5	-4	$\sqrt{\frac{7}{512\pi}}\sin^3\theta\, e^{-4i\varphi}$	$-5\sin\theta$	$4\cos\theta$	$-4i$

Explicit expressions for the components of $\vec{Y}_{j\ell m}(\Omega)$

$$\hat{r}\cdot\vec{Y}_{j\ell m}(\Omega) = \begin{cases} \sqrt{\dfrac{j}{2j+1}}\; Y_{jm} & (\ell = j-1) \\ 0 & (\ell = j) \\ -\sqrt{\dfrac{j+1}{2j+1}}\; Y_{jm} & (\ell = j+1) \end{cases}$$

$$\hat{\theta}\cdot\vec{Y}_{j\ell m}(\Omega) = \begin{cases} \sqrt{\dfrac{1}{j(2j+1)}}\; \dfrac{\partial}{\partial\theta}\, Y_{jm} & (\ell = j-1) \\ -\sqrt{\dfrac{1}{j(j+1)}}\; \dfrac{m}{\sin\theta}\, Y_{jm} & (\ell = j) \\ \sqrt{\dfrac{1}{(j+1)(2j+1)}}\; \dfrac{\partial}{\partial\theta}\, Y_{jm} & (\ell = j+1) \end{cases}$$

$$\hat{\varphi}\cdot\vec{Y}_{j\ell m}(\Omega)=\begin{cases}\sqrt{\dfrac{1}{j(2j+1)}}\;\dfrac{im}{\sin\theta}\;Y_{jm} & (\ell=j-1)\\[2ex] -\sqrt{\dfrac{1}{j(j+1)}}\;i\;\dfrac{\partial}{\partial\theta}\,Y_{jm} & (\ell=j)\\[2ex] \sqrt{\dfrac{1}{(j+1)(2j+1)}}\;\dfrac{im}{\sin\theta}\;Y_{jm} & (\ell=j+1)\end{cases}$$

5C. $\vec{Y}^{e}_{jm}$ and $\vec{Y}^{m}_{jm}$

The format is similar to that of the preceding section. At the end of this section we give explicit expressions for the individual components. Note the symmetries.

$$\vec{Y}^{e}_{j-m}= (-1)^{m}\,\vec{Y}^{e*}_{jm} \qquad , \qquad \vec{Y}^{m}_{j-m}= (-1)^{m+1}\,Y^{m*}_{jm}$$

			$\vec{Y}^{e*}_{jm}$		$\vec{Y}^{m}_{jm}$	
j	m	factor	$\hat{\theta}$	$\hat{\varphi}$	$\hat{\theta}$	$\hat{\varphi}$
1	1	$\sqrt{\frac{3}{16\pi}}\,e^{i\varphi}$	$-\cos\theta$	$-i$	1	$i\cos\theta$
1	0	$\sqrt{\frac{3}{8\pi}}\,\sin\theta$	-1	0	0	i
1	-1	$\sqrt{\frac{3}{16\pi}}\,e^{-i\varphi}$	$\cos\theta$	$-i$	1	$-i\cos\theta$
2	2	$\sqrt{\frac{5}{16\pi}}\,\sin\theta\, e^{2i\varphi}$	$\cos\theta$	i	-1	$-i\cos\theta$
2	1	$\sqrt{\frac{5}{16\pi}}\,e^{i\varphi}$	$1-2\cos^2\theta$	$-i\cos\theta$	$\cos\theta$	$i(2\cos^2\theta-1)$
2	0	$\sqrt{\frac{15}{8\pi}}\,\sin\theta$	$-\cos\theta$	0	0	$i\cos\theta$

			$\vec{Y}^{e*}_{jm}$		$\vec{Y}^{m}_{jm}$	
j	m	factor	$\hat{\theta}$	$\hat{\varphi}$	$\hat{\theta}$	$\hat{\varphi}$
2	-1	$\sqrt{\frac{5}{16\pi}}\, e^{-i\varphi}$	$2\cos^2\theta-1$	$-i\cos\theta$	$\cos\theta$	$i(1-2\cos^2\theta)$
2	-2	$\sqrt{\frac{5}{16\pi}}\sin\theta\, e^{-2i\varphi}$	$\cos\theta$	$-i$	1	$-i\cos\theta$
3	3	$\sqrt{\frac{105}{256\pi}}\sin^2\theta e^{3i\varphi}$	$-\cos\theta$	$-i$	1	$i\cos\theta$
3	2	$\sqrt{\frac{35}{128\pi}}\sin\theta\, e^{2i\varphi}$	$3\cos^2\theta-1$	$2i\cos\theta$	$-2\cos\theta$	$i(1-3\cos^2\theta)$
3	1	$\sqrt{\frac{7}{256\pi}}\, e^{i\varphi}$	$\cos\theta(11-15\cos^2\theta)$	$i(1-5\cos^2\theta)$	$5\cos^2\theta-1$	$i\cos\theta(15\cos^2\theta-11)$
3	0	$\sqrt{\frac{21}{64\pi}}\sin\theta$	$1-5\cos^2\theta$	0	0	$i(5\cos^2\theta-1)$
3	-1	$\sqrt{\frac{7}{256\pi}}\, e^{-i\varphi}$	$\cos\theta(15\cos^2\theta-11)$	$i(1-5\cos^2\theta)$	$5\cos^2\theta-1$	$i\cos\theta(11-15\cos^2\theta)$
3	-2	$\sqrt{\frac{35}{128\pi}}\sin\theta e^{-2i\varphi}$	$3\cos^2\theta-1$	$-2i\cos\theta$	$2\cos\theta$	$i(1-3\cos^2\theta)$
3	-3	$\sqrt{\frac{105}{256\pi}}\sin^2\theta e^{-3i\varphi}$	$\cos\theta$	$-i$	1	$-i\cos\theta$
4	4	$\sqrt{\frac{63}{128\pi}}\sin^3\theta e^{4i\varphi}$	$\cos\theta$	i	-1	$-i\cos\theta$
4	3	$\sqrt{\frac{63}{256\pi}}\sin^2\theta e^{3i\varphi}$	$1-4\cos^2\theta$	$-3i\cos\theta$	$3\cos\theta$	$i(4\cos^2\theta-1)$
4	2	$\sqrt{\frac{9}{128\pi}}\sin\theta e^{2i\varphi}$	$2\cos\theta(7\cos^2\theta-4)$	$i(7\cos^2\theta-1)$	$1-7\cos^2\theta$	$-2i\cos\theta(7\cos^2\theta-4)$
4	1	$\sqrt{\frac{9}{256\pi}}\, e^{i\varphi}$	$-3+27\cos^2\theta-28\cos^4\theta$	$i\cos\theta(3-7\cos^2\theta)$	$\cos\theta(7\cos^2\theta-3)$	$i(3-27\cos^2\theta+28\cos^4\theta)$
4	0	$\sqrt{\frac{45}{64\pi}}\sin\theta$	$\cos\theta(3-7\cos^2\theta)$	0	0	$i\cos\theta(7\cos^2\theta-3)$
4	-1	$\sqrt{\frac{9}{256\pi}}\, e^{-i\varphi}$	$3-27\cos^2\theta+28\cos^4\theta$	$i\cos\theta(3-7\cos^2\theta)$	$\cos\theta(7\cos^2\theta-3)$	$i(-3+27\cos^2\theta-28\cos^4\theta)$
4	-2	$\sqrt{\frac{9}{128\pi}}\sin\theta e^{-2i\varphi}$	$2\cos\theta(7\cos^2\theta-4)$	$i(1-7\cos^2\theta)$	$7\cos^2\theta-1$	$-2i\cos\theta(7\cos^2\theta-4)$
4	-3	$\sqrt{\frac{63}{256\pi}}\sin^2\theta e^{-3i\varphi}$	$4\cos^2\theta-1$	$-3i\cos\theta$	$3\cos\theta$	$i(1-4\cos^2\theta)$
4	-4	$\sqrt{\frac{63}{128\pi}}\sin^3\theta e^{-4i\varphi}$	$\cos\theta$	$-i$	1	$-i\cos\theta$

Explicit expressions for the components of $\vec{Y}^{e}_{jm}(\Omega)$ and $\vec{Y}^{m}_{jm}(\Omega)$:

$$\vec{Y}^{e}_{jm}(\Omega) = \sqrt{\frac{1}{j(j+1)}}\left[\left(\frac{\partial}{\partial\theta}Y_{jm}\right)\hat{\theta} + i\left(\frac{m}{\sin\theta}Y_{jm}\right)\hat{\varphi}\right]$$

$$\vec{Y}^{m}_{jm}(\Omega) = \sqrt{\frac{1}{j(j+1)}}\left[\left(-\frac{m}{\sin\theta}Y_{jm}\right)\hat{\theta} - i\left(\frac{\partial}{\partial\theta}Y_{jm}\right)\hat{\varphi}\right]$$

5D. $\underset{\sim}{T}_{j\ell m}$

For each $(j\ell m)$ we give an overall factor, and then the coefficients of $\hat{r}\hat{r}$, $\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$, $\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$, $\hat{\theta}\hat{\theta}$, $\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$ and $\hat{\varphi}\hat{\varphi}$ in the tensor harmonic. At the end of this section we give explicit expressions for these components. Note the symmetry

$$\underset{\sim}{T}_{j\ell -m} = (-)^{j+\ell+m}\,\underset{\sim}{T}^{*}_{j\ell m}$$

J	L	M	Factor	$\hat{r}\hat{r}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$	$\hat{\varphi}\hat{\varphi}$
0	2	0	$\sqrt{\frac{1}{24\pi}}$	2	0	0	-1	0	-1
1	1	1	$\sqrt{\frac{1}{160\pi}}\,e^{i\varphi}$	$4\sin\theta$	$3\cos\theta$	$3i$	$-2\sin\theta$	0	$-2\sin\theta$
1	1	0	$\sqrt{\frac{1}{80\pi}}$	$-4\cos\theta$	$3\sin\theta$	0	$2\cos\theta$	0	$2\cos\theta$
1	1	-1	$\sqrt{\frac{1}{160\pi}}\,e^{-i\varphi}$	$-4\sin\theta$	$-3\cos\theta$	$3i$	$2\sin\theta$	0	$2\sin\theta$

J	L	M	Factor	$\hat{r}\hat{r}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$	$\hat{\varphi}\hat{\varphi}$
1	2	1	$\sqrt{\frac{3}{32\pi}}\, e^{-i\varphi}$	0	-1	$-i \cos\theta$	0	0	0
1	2	0	$\sqrt{\frac{3}{16\pi}}$	0	0	$-i \sin\theta$	0	0	0
1	2	-1	$\sqrt{\frac{3}{32\pi}}\, e^{-i\varphi}$	0	-1	$i \cos\theta$	0	0	0
1	3	1	$\sqrt{\frac{3}{80\pi}}\, e^{i\varphi}$	$-2 \sin\theta$	$\cos\theta$	i	$\sin\theta$	0	$\sin\theta$
1	3	0	$\sqrt{\frac{3}{40\pi}}$	$2 \cos\theta$	$\sin\theta$	0	$-\cos\theta$	0	$-\cos\theta$
1	3	-1	$\sqrt{\frac{3}{80\pi}}\, e^{-i\varphi}$	$2 \sin\theta$	$-\cos\theta$	i	$-\sin\theta$	0	$-\sin\theta$
2	0	2	$\sqrt{\frac{1}{16\pi}}\, e^{2i\varphi}$	$\sin^2\theta$	$\sin\theta\cos\theta$	$i \sin\theta$	$\cos^2\theta$	$i \cos\theta$	-1
2	0	1	$\sqrt{\frac{1}{16\pi}}\, e^{i\varphi}$	$-2 \sin\theta\cos\theta$	$1-2\cos^2\theta$	$-i \cos\theta$	$2 \sin\theta\cos\theta$	$i \sin\theta$	0
2	0	0	$\sqrt{\frac{1}{24\pi}}$	$3\cos^2\theta-1$	$-3 \sin\theta\cos\theta$	0	$2-3\cos^2\theta$	0	-1
2	0	-1	$\sqrt{\frac{1}{16\pi}}\, e^{-i\varphi}$	$2 \sin\theta\cos\theta$	$2\cos^2\theta-1$	$-i \cos\theta$	$-2 \sin\theta\cos\theta$	$i \sin\theta$	0
2	0	-2	$\sqrt{\frac{1}{16\pi}}\, e^{-2i\varphi}$	$\sin^2\theta$	$\sin\theta\cos\theta$	$-i \sin\theta$	$\cos^2\theta$	$-i \cos\theta$	-1

J	L	M	Factor	$\hat{r}\hat{r}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$	$\hat{\varphi}\hat{\varphi}$
2	1	2	$\sqrt{\frac{1}{32\pi}}\,e^{2i\varphi}$	0	$-\sin\theta$	$-i\sin\theta\cos\theta$	$-2\cos\theta$	$-i(1+\cos^2\theta)$	$2\cos\theta$
2	1	1	$\sqrt{\frac{1}{32\pi}}\,e^{i\varphi}$	0	$\cos\theta$	$i(2\cos^2\theta-1)$	$-2\sin\theta$	$-2i\sin\theta\cos\theta$	$2\sin\theta$
2	1	0	$\sqrt{\frac{3}{16\pi}}$	0	0	$i\sin\theta\cos\theta$	0	$-i\sin^2\theta$	0
2	1	-1	$\sqrt{\frac{1}{32\pi}}\,e^{-i\varphi}$	0	$\cos\theta$	$-i(2\cos^2\theta-1)$	$-2\sin\theta$	$2i\sin\theta\cos\theta$	$2\sin\theta$
2	1	-2	$\sqrt{\frac{1}{32\pi}}\,e^{-2i\varphi}$	0	$\sin\theta$	$-i\sin\theta\cos\theta$	$2\cos\theta$	$-i(1+\cos^2\theta)$	$-2\cos\theta$
2	2	2	$\sqrt{\frac{5}{224\pi}}\,e^{2i\varphi}$	$-2\sin^2\theta$	$-\sin\theta\cos\theta$	$-i\sin\theta$	2	$2i\cos\theta$	$-2\cos^2\theta$
2	2	1	$\sqrt{\frac{5}{224\pi}}\,e^{i\varphi}$	$4\sin\theta\cos\theta$	$2\cos^2\theta-1$	$i\cos\theta$	0	$2i\sin\theta$	$-4\sin\theta\cos\theta$
2	2	0	$\sqrt{\frac{5}{336\pi}}$	$2-6\cos^2\theta$	$3\sin\theta\cos\theta$	0	2	0	$6\cos^2\theta-4$
2	2	-1	$\sqrt{\frac{5}{224\pi}}\,e^{-i\varphi}$	$-4\sin\theta\cos\theta$	$1-2\cos^2\theta$	$i\cos\theta$	0	$2i\sin\theta$	$4\sin\theta\cos\theta$
2	2	-2	$\sqrt{\frac{5}{224\pi}}\,e^{-2i\varphi}$	$-2\sin^2\theta$	$-\sin\theta\cos\theta$	$i\sin\theta$	2	$-2i\cos\theta$	$-2\cos^2\theta$

J	L	M	Factor	$\hat{r}\hat{r}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$	$\varphi\varphi$
2	3	2	$\sqrt{\frac{1}{128\pi}}\,e^{2i\varphi}$	0	$4\sin\theta$	$4i\sin\theta\cos\theta$	$-2\cos\theta$	$-i(1+\cos^2\theta)$	$2\cos\theta$
2	3	1	$\sqrt{\frac{1}{32\pi}}\,e^{i\varphi}$	0	$-2\cos\theta$	$2i(1-2\cos^2\theta)$	$-\sin\theta$	$-i\sin\theta\cos\theta$	$\sin\theta$
2	3	0	$\sqrt{\frac{3}{64\pi}}$	0	0	$-4i\sin\theta\cos\theta$	0	$-i\sin^2\theta$	0
2	3	-1	$\sqrt{\frac{1}{32\pi}}\,e^{-i\varphi}$	0	$-2\cos\theta$	$2i(2\cos^2\theta-1)$	$-\sin\theta$	$i\sin\theta\cos\theta$	$\sin\theta$
2	3	-2	$\sqrt{\frac{1}{128\pi}}\,e^{-2i\varphi}$	0	$-4\sin\theta$	$4i\sin\theta\cos\theta$	$2\cos\theta$	$-i(1+\cos^2\theta)$	$-2\cos\theta$
2	4	2	$\sqrt{\frac{1}{896\pi}}\,e^{2i\varphi}$	$12\sin^2\theta$	$-8\sin\theta\cos\theta$	$-8i\sin\theta$	$7\cos^2\theta-5$	$2i\cos\theta$	$5\cos^2\theta-7$
2	4	1	$\sqrt{\frac{1}{224\pi}}\,e^{i\varphi}$	$-12\sin\theta\cos\theta$	$8\cos^2\theta-4$	$4i\cos\theta$	$7\sin\theta\cos\theta$	$i\sin\theta$	$5\sin\theta\cos\theta$
2	4	0	$\sqrt{\frac{3}{448\pi}}$	$4(3\cos^2\theta-1)$	$8\sin\theta\cos\theta$	0	$-7\cos^2\theta+3$	0	$-5\cos^2\theta+1$
2	4	-1	$\sqrt{\frac{1}{224\pi}}\,e^{-i\varphi}$	$12\sin\theta\cos\theta$	$4-8\cos^2\theta$	$4i\cos\theta$	$-7\sin\theta\cos\theta$	$i\sin\theta$	$-5\sin\theta\cos\theta$
2	4	-2	$\sqrt{\frac{1}{896\pi}}\,e^{-2i\varphi}$	$12\sin^2\theta$	$-8\sin\theta\cos\theta$	$8i\sin\theta$	$7\cos^2\theta-5$	$-2i\cos\theta$	$5\cos^2\theta-7$

J	L	M	Factor	$\hat{r}\hat{r}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$	$\hat{\varphi}\hat{\varphi}$
3	1	3	$\sqrt{\frac{3}{32\pi}}\, e^{3i\varphi}$	$-\sin^3\theta$	$-\sin^2\theta\cos\theta$	$-i\sin^2\theta$	$-\sin\theta\cos^2\theta$	$-i\sin\theta\cos\theta$	$\sin\theta$
3	1	2	$\sqrt{\frac{1}{16\pi}}\, e^{2i\varphi}$	$3\sin^2\theta\cos\theta$	$\sin\theta(3\cos^2\theta-1)$	$2i\sin\theta\cos\theta$	$\cos\theta(3\cos^2\theta-2)$	$i(2\cos^2\theta-1)$	$-\cos\theta$
3	1	1	$\sqrt{\frac{1}{160\pi}}\, e^{i\varphi}$	$\sin\theta(3-15\cos^2\theta)$	$\cos\theta(11-15\cos^2\theta)$	$i(1-5\cos^2\theta)$	$\sin\theta(15\cos^2\theta-4)$	$5i\sin\theta\cos\theta$	$\sin\theta$
3	1	0	$\sqrt{\frac{3}{40\pi}}$	$\cos\theta(5\cos^2\theta-3)$	$\sin\theta(1-5\cos^2\theta)$	0	$\cos\theta\,(4-5\cos^2\theta)$	0	$-\cos\theta$
3	1	-1	$\sqrt{\frac{1}{160\pi}}\, e^{-i\varphi}$	$\sin\theta(15\cos^2\theta-3)$	$\cos\theta(15\cos^2\theta-11)$	$i(1-5\cos^2\theta)$	$\sin\theta(4-15\cos^2\theta)$	$5i\sin\theta\cos\theta$	$-\sin\theta$
3	1	-2	$\sqrt{\frac{1}{16\pi}}\, e^{-2i\varphi}$	$3\sin^2\theta\cos\theta$	$\sin\theta(3\cos^2\theta-1)$	$-2i\sin\theta\cos\theta$	$\cos\theta(3\cos^2\theta-2)$	$-i(2\cos^2\theta-1)$	$-\cos\theta$
3	1	-3	$\sqrt{\frac{3}{32\pi}}\, e^{-3i\varphi}$	$\sin^3\theta$	$\sin^2\theta\cos\theta$	$-i\sin^2\theta$	$\sin\theta\cos^2\theta$	$-i\sin\theta\cos\theta$	$-\sin\theta$
3	2	3	$\sqrt{\frac{15}{256\pi}}\, e^{3i\varphi}$	0	$\sin^2\theta$	$-i\sin^2\theta\cos\theta$	$2\sin\theta\cos\theta$	$i\sin\theta(1+\cos^2\theta)$	$-2\sin\theta\cos\theta$
3	2	2	$\sqrt{\frac{5}{128\pi}}\, e^{2i\varphi}$	0	$-2\sin\theta\cos\theta$	$i\sin\theta(1-3\cos^2\theta)$	$2-4\cos^2\theta$	$i\cos\theta(1-3\cos^2\theta)$	$4\cos^2\theta-2$

J	L	M	Factor	$\hat{r}\hat{r}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{\varphi}\hat{\varphi}$	$\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$	$\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$
3	2	1	$\sqrt{\frac{1}{256\pi}}\, e^{i\varphi}$	0	$5\cos^2\theta-1$	$-10\sin\theta\cos\theta$	$10\sin\theta\cos\theta$	$i\cos\theta(15\cos^2\theta-11)$	$i\sin\theta(5-15\cos^2\theta)$
3	2	0	$\sqrt{\frac{3}{64\pi}}$	0	0	0	0	$i\sin\theta(5\cos^2\theta-1)$	$-5i\sin^2\theta\cos\theta$
3	2	-1	$\sqrt{\frac{1}{256\pi}}\, e^{-i\varphi}$	0	$5\cos^2\theta-1$	$-10\sin\theta\cos\theta$	$10\sin\theta\cos\theta$	$i\cos\theta(11-15\cos^2\theta)$	$i\sin\theta(15\cos^2\theta-5)$
3	2	-2	$\sqrt{\frac{5}{128\pi}}\, e^{-2i\varphi}$	0	$2\sin\theta\cos\theta$	$4\cos^2\theta-2$	$2-4\cos^2\theta$	$i\sin\theta(1-3\cos^2\theta)$	$i\cos\theta(1-3\cos^2\theta)$
3	2	-3	$\sqrt{\frac{15}{256\pi}}\, e^{-3i\varphi}$	0	$\sin^2\theta$	$2\sin\theta\cos\theta$	$-2\sin\theta\cos\theta$	$-i\sin^2\theta\cos\theta$	$-i\sin\theta(1+\cos^2\theta)$
3	3	3	$\sqrt{\frac{7}{4608\pi}}\, e^{3i\varphi}$	$8\sin^3\theta$	$3\sin^2\theta\cos\theta$	$-2\sin\theta(\cos^2\theta+5)$	$2\sin\theta(5\cos^2\theta+1)$	$3i\sin^2\theta$	$-12i\sin\theta\cos\theta$
3	3	2	$\sqrt{\frac{7}{768\pi}}\, e^{2i\varphi}$	$-8\sin^2\theta\cos\theta$	$\sin\theta(1-3\cos^2\theta)$	$2\cos\theta(1+\cos^2\theta)$	$-2\cos\theta(5\cos^2\theta-3)$	$-2i\sin\theta\cos\theta$	$4i(2\cos^2\theta-1)$
3	3	1	$\sqrt{\frac{7}{7680\pi}}\, e^{i\varphi}$	$8\sin\theta(5\cos^2\theta-1)$	$\cos\theta(15\cos^2\theta-11)$	$2\sin\theta(5\cos^2\theta-3)$	$2\sin\theta(7-25\cos^2\theta)$	$i(5\cos^2\theta-1)$	$20i\sin\theta\cos\theta$
3	3	0	$\sqrt{\frac{7}{5760\pi}}$	$8\cos\theta(3-5\cos^2\theta)$	$3\sin\theta(5\cos^2\theta-1)$	$2\cos\theta(9-5\cos^2\theta)$	$2\cos\theta(25\cos^2\theta-21)$	0	0

J L M	Factor	$\hat{r}\hat{r}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$	$\hat{\varphi}\hat{\varphi}$
3 3 -1	$\sqrt{\frac{7}{7680\pi}}\,e^{-i\varphi}$	$8\sin\theta(1-5\cos^2\theta)$	$\cos\theta(11-15\cos^2\theta)$	$i(5\cos^2\theta-1)$	$2\sin\theta(3-5\cos^2\theta)$	$20\,i\sin\theta\cos\theta$	$2\sin\theta(25\cos^2\theta-7)$
3 3 -2	$\sqrt{\frac{7}{768\pi}}\,e^{-2i\varphi}$	$-8\sin^2\theta\cos\theta$	$\sin\theta(1-3\cos^2\theta)$	$2\,i\sin\theta\cos\theta$	$2\cos\theta(1+\cos^2\theta)$	$4\,i(1-2\cos^2\theta)$	$-2\cos\theta(5\cos^2\theta-3)$
3 3 -3	$\sqrt{\frac{7}{4608\pi}}\,e^{-3i\varphi}$	$-8\sin^3\theta$	$-3\sin^2\theta\cos\theta$	$3\,i\sin^2\theta$	$2\sin\theta(\cos^2\theta+5)$	$-12\,i\sin\theta\cos\theta$	$-2\sin\theta(5\cos^2\theta+1)$
3 4 3	$\sqrt{\frac{3}{512\pi}}\,e^{3i\varphi}$	0	$-5\sin^2\theta$	$-5\,i\sin^2\theta\cos\theta$	$4\sin\theta\cos\theta$	$2\,i\sin\theta(1+\cos^2\theta)$	$-4\sin\theta\cos\theta$
3 4 2	$\sqrt{\frac{1}{256\pi}}\,e^{2i\varphi}$	0	$10\sin\theta\cos\theta$	$5\,i\sin\theta(3\cos^2\theta-1)$	$4(1-2\cos^2\theta)$	$2\,i\cos\theta(1-3\cos^2\theta)$	$4(2\cos^2\theta-1)$
3 4 1	$\sqrt{\frac{5}{512\pi}}\,e^{i\varphi}$	0	$1-5\cos^2\theta$	$i\cos\theta(11-15\cos^2\theta)$	$-4\sin\theta\cos\theta$	$2\,i\sin\theta(1-3\cos^2\theta)$	$4\sin\theta\cos\theta$
3 4 0	$\sqrt{\frac{15}{128\pi}}$	0	0	$i\sin\theta(1-5\cos^2\theta)$	0	$-2\,i\sin^2\theta\cos\theta$	0
3 4 -1	$\sqrt{\frac{5}{512\pi}}\,e^{-i\varphi}$	0	$1-5\cos^2\theta$	$i\cos\theta(15\cos^2\theta-11)$	$-4\sin\theta\cos\theta$	$2\,i\sin\theta(3\cos^2\theta-1)$	$4\sin\theta\cos\theta$
3 4 -2	$\sqrt{\frac{1}{256\pi}}\,e^{-2i\varphi}$	0	$-10\sin\theta\cos\theta$	$5\,i\sin\theta(3\cos^2\theta-1)$	$4(2\cos^2\theta-1)$	$2\,i\cos\theta(1-3\cos^2\theta)$	$4(1-2\cos^2\theta)$

J	L	M	Factor	$\hat{r}\hat{r}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$	$\hat{\varphi}\hat{\varphi}$
3	4	-3	$\sqrt{\frac{3}{512\pi}}\,e^{-3i\varphi}$	0	$-5\sin^2\theta$	$5i\sin^2\theta\cos\theta$	$4\sin\theta\cos\theta$	$-2i\sin\theta(1+\cos^2\theta)$	$-4\sin\theta\cos\theta$
3	5	3	$\sqrt{\frac{1}{2304\pi}}\,e^{3i\varphi}$	$-20\sin^3\theta$	$15\sin^2\theta\cos\theta$	$15i\sin^2\theta$	$\sin\theta(7-13\cos^2\theta)$	$-6i\sin\theta\cos\theta$	$\sin\theta(13-7\cos^2\theta)$
3	5	2	$\sqrt{\frac{1}{348\pi}}\,e^{2i\varphi}$	$20\sin^2\theta\cos\theta$	$5\sin\theta(1-3\cos^2\theta)$	$-10i\sin\theta\cos\theta$	$\cos\theta(13\cos^2\theta-11)$	$2i(2\cos^2\theta-1)$	$\cos\theta(7\cos^2\theta-9)$
3	5	1	$\sqrt{\frac{5}{768\pi}}\,e^{i\varphi}$	$4\sin\theta(1-5\cos^2\theta)$	$\cos\theta(15\cos^2\theta-11)$	$i(5\cos^2\theta-1)$	$\sin\theta(13\cos^2\theta-3)$	$2i\sin\theta\cos\theta$	$\sin\theta(7\cos^2\theta-1)$
3	5	0	$\sqrt{\frac{5}{576\pi}}$	$4\cos\theta(5\cos^2\theta-3)$	$3\sin\theta(5\cos^2\theta-1)$	0	$\cos\theta(9-13\cos^2\theta)$	0	$\cos\theta(3-7\cos^2\theta)$
3	5	-1	$\sqrt{\frac{5}{768\pi}}\,e^{-i\varphi}$	$4\sin\theta(5\cos^2\theta-1)$	$\cos\theta(11-15\cos^2\theta)$	$i(5\cos^2\theta-1)$	$\sin\theta(3-13\cos^2\theta)$	$2i\sin\theta\cos\theta$	$\sin\theta(1-7\cos^2\theta)$
3	5	-2	$\sqrt{\frac{1}{384\pi}}\,e^{-2i\varphi}$	$20\sin^2\theta\cos\theta$	$5\sin\theta(1-3\cos^2\theta)$	$10i\sin\theta\cos\theta$	$\cos\theta(13\cos^2\theta-11)$	$2i(1-2\cos^2\theta)$	$\cos\theta(7\cos^2\theta-9)$
3	5	-3	$\sqrt{\frac{1}{2304\pi}}\,e^{-3i\varphi}$	$20\sin^3\theta$	$-15\sin^2\theta\cos\theta$	$15i\sin^2\theta$	$\sin\theta(13\cos^2\theta-7)$	$-6i\sin\theta\cos\theta$	$\sin\theta(7\cos^2\theta-13)$
4	2	4	$\sqrt{\frac{15}{128\pi}}\,e^{4i\varphi}$	$\sin^4\theta$	$\sin^3\theta\cos\theta$	$i\sin^3\theta$	$\sin^2\theta\cos^2\theta$	$i\sin^2\theta\cos\theta$	$-\sin^2\theta$

J	L	M	Factor	$\hat{r}\hat{r}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$	$\hat{\varphi}\hat{\varphi}$
4	2	3	$\sqrt{\frac{15}{256\pi}}\,e^{3i\varphi}$	$-4\sin^3\theta\cos\theta$	$\sin^2\theta(1-4\cos^2\theta)$	$-3i\sin^2\theta\cos\theta$	$2\sin\theta\cos\theta(1-2\cos^2\theta)$	$i\sin\theta(1-3\cos^2\theta)$	$2\sin\theta\cos\theta$
4	2	2	$\sqrt{\frac{15}{896\pi}}\,e^{2i\varphi}$	$2\sin^2\theta(7\cos^2\theta-1)$	$2\sin\theta\cos\theta(7\cos^2\theta-4)$	$i\sin\theta(7\cos^2\theta-1)$	$2(7\cos^4\theta-7\cos^2\theta+1)$	$i\cos\theta(7\cos^2\theta-5)$	$-2\cos^2\theta$
4	2	1	$\sqrt{\frac{15}{1792\pi}}\,e^{i\varphi}$	$4\sin\theta\cos\theta(3-7\cos^2\theta)$	$-3+27\cos^2\theta-28\cos^4\theta$	$i\cos\theta(3-7\cos^2\theta)$	$14\sin\theta\cos\theta(2\cos^2\theta-1)$	$i\sin\theta(7\cos^2\theta-1)$	$2\sin\theta\cos\theta$
4	2	0	$\sqrt{\frac{3}{448\pi}}$	$35\cos^4\theta-30\cos^2\theta+3$	$5\sin\theta\cos\theta(3-7\cos^2\theta)$	0	$-4+35\cos^2\theta-35\cos^4\theta$	0	$1-5\cos^2\theta$
4	2	-1	$\sqrt{\frac{15}{1792\pi}}\,e^{-i\varphi}$	$4\sin\theta\cos\theta(7\cos^2\theta-3)$	$3-27\cos^2\theta+28\cos^4\theta$	$i\cos\theta(3-7\cos^2\theta)$	$14\sin\theta\cos\theta(1-2\cos^2\theta)$	$i\sin\theta(7\cos^2\theta-1)$	$-2\sin\theta\cos\theta$
4	2	-2	$\sqrt{\frac{15}{896\pi}}\,e^{-2i\varphi}$	$2\sin^2\theta(7\cos^2\theta-1)$	$2\sin\theta\cos\theta(7\cos^2\theta-4)$	$i\sin\theta(1-7\cos^2\theta)$	$2(7\cos^4\theta-7\cos^2\theta+1)$	$i\cos\theta(5-7\cos^2\theta)$	$-2\cos^2\theta$
4	2	-3	$\sqrt{\frac{15}{256\pi}}\,e^{-3i\varphi}$	$4\sin^3\theta\cos\theta$	$\sin^2\theta(4\cos^2\theta-1)$	$-3i\sin^2\theta\cos\theta$	$2\sin\theta\cos\theta(2\cos^2\theta-1)$	$i\sin\theta(1-3\cos^2\theta)$	$-2\sin\theta\cos\theta$
4	2	-4	$\sqrt{\frac{15}{128\pi}}\,e^{-4i\varphi}$	$\sin^4\theta$	$\sin^3\theta\cos\theta$	$-i\sin^3\theta$	$\sin^2\theta\cos^2\theta$	$-i\sin^2\theta\cos$	$-\sin^2\theta$
4	3	4	$\sqrt{\frac{21}{256\pi}}\,e^{4i\varphi}$	0	$-\sin^3\theta$	$-i\sin^3\theta\cos\theta$	$-2\sin^2\theta\cos\theta$	$-i\sin^2\theta(1+\cos^2\theta)$	$2\sin^2\theta\cos\theta$

J	L	M	Factor	$\hat{r}\hat{r}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$	$\hat{\varphi}\hat{\varphi}$	$\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$
4	3	3	$\sqrt{\frac{21}{512\pi}}\,e^{3i\varphi}$	0	$3\sin^2\theta\cos\theta$	$2\sin\theta(3\cos^2\theta-1)$	$i\sin^2\theta(4\cos^2\theta-1)$	$2\sin\theta(1-3\cos^2\theta)$	$4i\sin\theta\cos^3\theta$
4	3	2	$\sqrt{\frac{3}{256\pi}}\,e^{2i\varphi}$	0	$\sin\theta(1-7\cos^2\theta)$	$2\cos\theta(5-7\cos^2\theta)$	$-2i\sin\theta\cos\theta(7\cos^2\theta-4)$	$2\cos\theta(7\cos^2\theta-5)$	$2i(-7\cos^4\theta+6\cos^2\theta-1)$
4	3	1	$\sqrt{\frac{3}{512\pi}}\,e^{i\varphi}$	0	$\cos\theta(7\cos^2\theta-3)$	$2\sin\theta(1-7\cos^2\theta)$	$i(3-27\cos^2\theta+28\cos^4\theta)$	$2\sin\theta(7\cos^2\theta-1)$	$4i\sin\theta\cos\theta(4-7\cos^2\theta)$
4	3	0	$\sqrt{\frac{15}{512\pi}}$	0	0	0	$2i\sin\theta\cos\theta\,(7\cos^2\theta-3)$	0	$2i\sin^2\theta(1-7\cos^2\theta)$
4	3	-1	$\sqrt{\frac{3}{512\pi}}\,e^{-i\varphi}$	0	$\cos\theta(7\cos^2\theta-3)$	$2\sin\theta(1-7\cos^2\theta)$	$i(-3+27\cos^2\theta-28\cos^4\theta)$	$2\sin\theta(7\cos^2\theta-1)$	$4i\sin\theta\cos\theta(7\cos^2\theta-4)$
4	3	-2	$\sqrt{\frac{3}{256\pi}}\,e^{-2i\varphi}$	0	$\sin\theta(7\cos^2\theta-1)$	$2\cos\theta(7\cos^2\theta-5)$	$-2i\sin\theta\cos\theta(7\cos^2\theta-4)$	$2\cos\theta(5-7\cos^2\theta)$	$2i(-7\cos^4\theta+6\cos^2\theta-1)$
4	3	-3	$\sqrt{\frac{21}{512\pi}}\,e^{-3i\varphi}$	0	$3\sin^2\theta\cos\theta$	$2\sin\theta(3\cos^2\theta-1)$	$i\sin^2\theta(1-4\cos^2\theta)$	$2\sin\theta(1-3\cos^2\theta)$	$-4i\sin\theta\cos^3\theta$
4	3	-4	$\sqrt{\frac{21}{256\pi}}\,e^{-4i\varphi}$	0	$\sin^3\theta$	$2\sin^2\theta\cos\theta$	$-i\sin^3\theta\cos\theta$	$-2\sin^2\theta\cos\theta$	$-i\sin^2\theta(1+\cos^2\theta)$
4	4	4	$\sqrt{\frac{3}{2816\pi}}\,e^{4i\varphi}$	$-10\sin^4\theta$	$-3\sin^3\theta\cos\theta$	$2\sin^2\theta(2\cos^2\theta+7)$	$-3i\sin^3\theta$	$-2\sin^2\theta(7\cos^2\theta+2)$	$18i\sin^2\theta\cos\theta$

J	L	M	Factor	$\hat{r}\hat{r}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$	$\hat{\varphi}\hat{\varphi}$
4	4	3	$\sqrt{\frac{3}{5632\pi}}\,e^{3i\varphi}$	$40\sin^3\theta\cos\theta$	$3\sin^2\theta(4\cos^2\theta-1)$	$9\,i\sin^2\theta\cos\theta$	$-4\sin\theta\cos\theta(4\cos^2+5)$	$18\,i\sin\theta(1-3\cos^2\theta)$	$4\sin\theta\cos\theta(14\cos^2\theta-5)$
4	4	2	$\sqrt{\frac{3}{19712\pi}}\,e^{2i\varphi}$	$20\sin^2\theta(1-7\cos^2\theta)$	$6\sin\theta\cos\theta(4-7\cos^2\theta)$	$3\,i\sin\theta(1-7\cos^2\theta)$	$4(14\cos^4\theta-7\cos^2\theta+2)$	$18\,i\cos\theta(7\cos^2\theta-5)$	$4(-49\cos^4\theta+47\cos^2\theta-7)$
4	4	1	$\sqrt{\frac{3}{39424\pi}}\,e^{i\varphi}$	$40\sin\theta\cos\theta(7\cos^2\theta-3)$	$3(3-27\cos^2\theta+28\cos^4\theta)$	$3\,i\cos\theta(7\cos^2\theta-3)$	$28\sin\theta\cos\theta(4\cos^2\theta-3)$	$18\,i\sin\theta(7\cos^2\theta-1)$	$4\sin\theta\cos\theta(51-98\cos^2\theta)$
4	4	0	$\sqrt{\frac{15}{9856\pi}}$	$2(-35\cos^4\theta+30\cos^2\theta-3)$	$3\sin\theta\cos\theta(7\cos^2\theta-3)$	0	$2(-3+21\cos^2\theta-14\cos^4\theta)$	0	$2(6-51\cos^2\theta+49\cos^4\theta)$
4	4	-1	$\sqrt{\frac{3}{39424\pi}}\,e^{-i\varphi}$	$40\sin\theta\cos\theta(3-7\cos^2\theta)$	$3(-3+27\cos^2\theta-28\cos^4\theta)$	$3i\cos\theta(7\cos^2\theta-3)$	$28\sin\theta\cos\theta(3-4\cos^2\theta)$	$18\,i\sin\theta(7\cos^2\theta-1)$	$4\sin\theta\cos\theta(98\cos^2\theta-51)$
4	4	-2	$\sqrt{\frac{3}{19712\pi}}\,e^{-2i\varphi}$	$20\sin^2\theta(1-7\cos^2\theta)$	$6\sin\theta\cos\theta(4-7\cos^2\theta)$	$3i\sin\theta(7\cos^2\theta-1)$	$4(14\cos^4\theta-7\cos^2\theta+2)$	$18\,i\cos\theta(5-7\cos^2\theta)$	$4(-49\cos^4\theta+47\cos^2\theta-7)$
4	4	-3	$\sqrt{\frac{3}{5632\pi}}\,e^{-3i\varphi}$	$-40\sin^3\theta\cos\theta$	$3\sin^2\theta(1-4\cos^2\theta)$	$9\,i\sin^2\theta\cos\theta$	$4\sin\theta\cos\theta(4\cos^2\theta+5)$	$18\,i\sin\theta(1-3\cos^2\theta)$	$4\sin\theta\cos\theta(5-14\cos^2\theta)$
4	4	-4	$\sqrt{\frac{3}{2816\pi}}\,e^{-4i\varphi}$	$-10\sin^4\theta$	$-3\sin^3\theta\cos\theta$	$3\,i\sin^3\theta$	$2\sin\theta(2\cos^2\theta+7)$	$-18\,i\sin^2\theta\cos\theta$	$-2\sin^2\theta(7\cos^2\theta+2)$
4	5	4	$\sqrt{\frac{21}{2048\pi}}\,e^{4i\varphi}$	0	$4\sin^3\theta$	$4\,i\sin^3\theta\cos\theta$	$-4\sin^2\theta\cos\theta$	$-2\,i\sin^2\theta(1+\cos^2\theta)$	$4\sin^2\theta\cos\theta$

J	L	M	Factor	$\hat r\hat\theta+\hat\theta\hat r$	$\hat\theta\hat\theta$	$\hat\varphi\hat\varphi$
				ff	$\hat r\hat\varphi+\hat\varphi\hat r$	$\hat\theta\hat\varphi+\hat\varphi\hat\theta$
4	5	3	$\sqrt{\frac{21}{256\pi}}e^{3i\varphi}$	$-3\sin^2\theta\cos\theta$	$\sin\theta(3\cos^2\theta-1)$	$\sin\theta(1-3\cos^2\theta)$
				0	$i\sin^2\theta(1-4\cos^2\theta)$	$2i\sin\theta\cos^3\theta$
4	5	2	$\sqrt{\frac{3}{512\pi}}e^{2i\varphi}$	$2\sin\theta(7\cos^2\theta-1)$	$2\cos\theta(5-7\cos^2\theta)$	$2\cos\theta(7\cos^2\theta-5)$
				0	$4i\sin\theta\cos\theta(7\cos^2\theta-4)$	$2i(-1+6\cos^2\theta-7\cos^4\theta)$
4	5	1	$\sqrt{\frac{3}{256\pi}}e^{i\varphi}$	$\cos\theta(3-7\cos^2\theta)$	$\sin\theta(1-7\cos^2\theta)$	$\sin\theta(7\cos^2\theta-1)$
				0	$i(-3+27\cos^2\theta-28\cos^4\theta)$	$2i\sin\theta\cos\theta(4-7\cos^2\theta)$
4	5	0	$\sqrt{\frac{3}{5120\pi}}$	0	0	0
				0	$20i\sin\theta\cos\theta(3-7\cos^2\theta)$	$10i\sin^2\theta(1-7\cos^2\theta)$
4	5	-1	$\sqrt{\frac{3}{256\pi}}e^{-i\varphi}$	$\cos\theta(3-7\cos^2\theta)$	$\sin\theta(1-7\cos^2\theta)$	$\sin\theta(7\cos^2\theta-1)$
				0	$i(3-27\cos^2\theta+28\cos^4\theta)$	$2i\sin\theta\cos\theta(7\cos^2\theta-4)$
4	5	-2	$\sqrt{\frac{3}{512\pi}}e^{-2i\varphi}$	$2\sin\theta(1-7\cos^2\theta)$	$2\cos\theta(7\cos^2\theta-5)$	$2\cos\theta(5-7\cos^2\theta)$
				0	$4i\sin\theta\cos\theta(7\cos^2\theta-4)$	$2i(-1+6\cos^2\theta-7\cos^4\theta)$
4	5	-3	$\sqrt{\frac{21}{256\pi}}e^{-3i\varphi}$	$-3\sin^2\theta\cos\theta$	$\sin\theta(3\cos^2\theta-1)$	$\sin\theta(1-3\cos^2\theta)$
				0	$i\sin^2\theta(4\cos^2\theta-1)$	$-2i\sin\theta\cos^3\theta$
4	5	-4	$\sqrt{\frac{21}{2048\pi}}e^{-4i\varphi}$	$-4\sin^3\theta$	$4\sin^2\theta\cos\theta$	$-4\sin^2\theta\cos\theta$
				0	$4i\sin^3\theta\cos\theta$	$-2i\sin^2\theta(1+\cos^2\theta)$
4	6	4	$\sqrt{\frac{21}{11264\pi}}e^{4i\varphi}$	$-8\sin^3\theta\cos\theta$	$\sin^2\theta(7\cos^2\theta-3)$	$\sin^2\theta(3\cos^2\theta-7)$
				$10\sin^4\theta$	$-8i\sin^3\theta$	$4i\sin^2\theta\cos\theta$

J	L	M	Factor	$\hat{r}\hat{r}$	$\hat{r}\hat{\theta}+\hat{\theta}\hat{r}$	$\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}$	$\hat{\theta}\hat{\theta}$	$\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}$	$\hat{\varphi}\hat{\varphi}$
4	6	3	$\sqrt{\frac{21}{1408\pi}}\,e^{3i\varphi}$	$-10\sin^3\theta\cos\theta$	$2\sin^2\theta(4\cos^2\theta-1)$	$6i\sin^2\theta\cos\theta$	$\sin\theta\cos\theta(5-7\cos^2\theta)$	$i\sin\theta(1-3\cos^2\theta)$	$\sin\theta\cos\theta(5-3\cos^2\theta)$
4	6	2	$\sqrt{\frac{3}{2816\pi}}\,e^{2i\varphi}$	$10\sin^2\theta(7\cos^2\theta-1)$	$8\sin\theta\cos\theta(4-7\cos^2\theta)$	$4i\sin\theta(1-7\cos^2\theta)$	$49\cos^4\theta-52\cos^2\theta+7$	$2i\cos\theta(7\cos^2\theta-5)$	$21\cos^4\theta-28\cos^2\theta+3$
4	6	1	$\sqrt{\frac{3}{1408\pi}}\,e^{i\varphi}$	$10\sin\theta\cos\theta(3-7\cos^2\theta)$	$2(3-27\cos^2\theta+28\cos^4\theta)$	$2i\cos\theta(7\cos^2\theta-3)$	$\sin\theta\cos\theta(49\cos^2\theta-23)$	$i\sin\theta(7\cos^2\theta-1)$	$7\sin\theta\cos\theta(3\cos^2\theta-1)$
4	6	0	$\sqrt{\frac{3}{28160\pi}}$	$10(35\cos^4\theta-30\cos^2\theta+3)$	$40\sin\theta\cos\theta(7\cos^2\theta-3)$	0	$5(-5+46\cos^2\theta-49\cos^4\theta)$	0	$5(-1+14\cos^2\theta-21\cos^4\theta)$
4	6	-1	$\sqrt{\frac{3}{1408\pi}}\,e^{-i\varphi}$	$10\sin\theta\cos\theta(7\cos^2\theta-3)$	$2(-3+27\cos^2\theta-28\cos^4\theta)$	$2i\cos\theta(7\cos^3\theta-3)$	$\sin\theta\cos\theta(23-49\cos^2\theta)$	$i\sin\theta(7\cos^2\theta-1)$	$7\sin\theta\cos\theta(1-3\cos^2\theta)$
4	6	-2	$\sqrt{\frac{3}{2816\pi}}\,e^{-2i\varphi}$	$10\sin^2\theta(7\cos^2\theta-1)$	$8\sin\theta\cos\theta(4-7\cos^2\theta)$	$4i\sin\theta(7\cos^2\theta-1)$	$49\cos^4\theta-52\cos^2\theta+7$	$2i\cos\theta(5-7\cos^2\theta)$	$21\cos^4\theta-28\cos^2\theta+3$
4	6	-3	$\sqrt{\frac{21}{1408\pi}}\,e^{-3i\varphi}$	$10\sin^3\theta\cos\theta$	$2\sin^2\theta(1-4\cos^2\theta)$	$6i\sin^2\theta\cos\theta$	$\sin\theta\cos\theta(7\cos^2\theta-5)$	$i\sin\theta(1-3\cos^2\theta)$	$\sin\theta\cos\theta(3\cos^2\theta-5)$
4	6	-4	$\sqrt{\frac{21}{11264\pi}}\,e^{-4i\varphi}$	$10\sin^4\theta$	$-8\sin^3\theta\cos\theta$	$8i\sin^3\theta$	$\sin^2\theta(7\cos^2\theta-3)$	$-4i\sin^2\theta\cos\theta$	$\sin^2\theta(3\cos^2\theta-7)$

Explicit expressions for the components of $\underset{\sim}{T}_{j\ell m}$

$$\hat{r}\hat{r}:\underset{\sim}{T}_{j\ell m}=\begin{cases}\sqrt{\dfrac{j(j-1)}{(2j+1)(2j-1)}}\;Y_{jm} & (\ell=j-2)\\[2ex] 0 & (\ell=j-1)\\[2ex] -\sqrt{\dfrac{2j(j+1)}{3(2j-1)(2j+3)}}\;Y_{jm} & (\ell=j)\\[2ex] 0 & (\ell=j+1)\\[2ex] \sqrt{\dfrac{(j+1)(j+2)}{(2j+1)(2j+3)}} & (\ell=j+2)\end{cases}$$

$$\frac{\hat{r}\hat{\theta}+\hat{\theta}\hat{r}}{2}:\underset{\sim}{T}_{j\ell m}=\begin{cases}\sqrt{\dfrac{j-1}{j(2j+1)(2j-1)}}\;\dfrac{\partial}{\partial\theta}\;Y_{jm} & (\ell=j-2)\\[2ex] -\sqrt{\dfrac{j-1}{2j(j+1)(2j+1)}}\;\dfrac{m}{\sin\theta}\;Y_{jm} & (\ell=j-1)\\[2ex] -\sqrt{\dfrac{3}{2j(j+1)(2j+3)(2j-1)}}\;\dfrac{\partial}{\partial\theta}\;Y_{jm} & (\ell=j)\\[2ex] \sqrt{\dfrac{j+2}{2j(j+1)(2j+1)}}\;\dfrac{m}{\sin\theta}\;Y_{jm} & (\ell=j+1)\\[2ex] -\sqrt{\dfrac{j+2}{(j+1)(2j+1)(2j+3)}}\;\dfrac{\partial}{\partial\theta}\;Y_{jm} & (\ell=j+2)\end{cases}$$

$$\frac{\hat{r}\hat{\varphi}+\hat{\varphi}\hat{r}}{2}:\underset{\sim}{T}_{j\ell m}=\begin{cases}\sqrt{\dfrac{j-1}{j(2j+1)(2j-1)}}\;\dfrac{im}{\sin\theta}\,Y_{jm} & (\ell=j-2)\\[2ex] -\sqrt{\dfrac{j-1}{2j(j+1)(2j+1)}}\;i\,\dfrac{\partial}{\partial\theta}\,Y_{jm} & (\ell=j-1)\\[2ex] -\sqrt{\dfrac{3}{2j(j+1)(2j+3)(2j-1)}}\;\dfrac{im}{\sin\theta}\,Y_{jm} & (\ell=j)\\[2ex] \sqrt{\dfrac{j+2}{2j(j+1)(2j+1)}}\;i\,\dfrac{\partial}{\partial\theta}\,Y_{jm} & (\ell=j+1)\\[2ex] -\sqrt{\dfrac{j+2}{(j+1)(2j+1)(2j+3)}}\;\dfrac{im}{\sin\theta}\,Y_{jm} & (\ell=j+2)\end{cases}$$

$$\hat{\theta}\hat{\theta}:\underset{\sim}{T}_{j\ell m}=\begin{cases}\sqrt{\dfrac{1}{j(j-1)(2j+1)(2j-1)}}\left(\dfrac{\partial^2}{\partial\theta^2}+j\right)Y_{jm} & (\ell=j-2)\\[2ex] -\sqrt{\dfrac{2}{j(j+1)(j-1)(2j+1)}}\;m\,\dfrac{\partial}{\partial\theta}\left(\dfrac{Y_{jm}}{\sin\theta}\right) & (\ell=j-1)\\[2ex] \sqrt{\dfrac{2}{3j(j+1)(2j-1)(2j+3)}}\left[3\,\dfrac{\partial^2}{\partial\theta^2}+2j(j+1)\right]Y_{jm} & (\ell=j)\\[2ex] -\sqrt{\dfrac{2}{j(j+1)(j+2)(2j+1)}}\;m\,\dfrac{\partial}{\partial\theta}\left(\dfrac{Y_{jm}}{\sin\theta}\right) & (\ell=j+1)\\[2ex] \sqrt{\dfrac{1}{(j+1)(j+2)(2j+1)(2j+3)}}\left[\dfrac{\partial^2}{\partial\theta^2}-(j+1)\right]Y_{jm} & (\ell=j+2)\end{cases}$$

$$\frac{\hat{\theta}\hat{\varphi}+\hat{\varphi}\hat{\theta}}{2} : \underset{\sim}{T}_{j\,\ell m} = \begin{cases} \sqrt{\dfrac{1}{j(j-1)(2j+1)(2j-1)}}\; i\,m \dfrac{\partial}{\partial\theta}\left(\dfrac{Y_{jm}}{\sin\theta}\right) & (\ell = j-2) \\[2ex] -\sqrt{\dfrac{1}{2j(j+1)(j-1)(2j+1)}}\; i\left[2\dfrac{\partial^2}{\partial\theta^2} + j(j+1)\right] Y_{jm} & (\ell = j-1) \\[2ex] \sqrt{\dfrac{6}{j(j+1)(2j-1)(2j+3)}}\; i\,m \dfrac{\partial}{\partial\theta}\left(\dfrac{Y_{jm}}{\sin\theta}\right) & (\ell = j) \\[2ex] -\sqrt{\dfrac{1}{2j(j+1)(j+2)(2j+1)}}\; i\left[2\dfrac{\partial^2}{\theta\,\theta^2} + j(j+1)\right] Y_{jm} & (\ell = j+1) \\[2ex] \sqrt{\dfrac{1}{(j+1)(j+2)(2j+1)(2j+3)}}\; i\,m \dfrac{\partial}{\partial\theta}\left(\dfrac{Y_{jm}}{\sin\theta}\right) & (\ell = j+2) \end{cases}$$

$$\hat{\varphi}\hat{\varphi} : \underset{\sim}{T}_{j\,\ell m} = \begin{cases} -\sqrt{\dfrac{1}{j(j-1)(2j+1)(2j-1)}}\left(\dfrac{\partial^2}{\partial\theta^2} + j^2\right) Y_{jm} & (\ell = j-2) \\[2ex] \sqrt{\dfrac{2}{j(j+1)(j-1)(2j+1)}}\; m \dfrac{\partial}{\partial\theta}\left(\dfrac{Y_{jm}}{\sin\theta}\right) & (\ell = j-1) \\[2ex] -\sqrt{\dfrac{2}{3j(j+1)(2j-1)(2j+3)}}\left[3\dfrac{\partial^2}{\partial\theta^2} + j(j+1)\right] Y_{jm} & (\ell = j) \\[2ex] \sqrt{\dfrac{2}{j(j+1)(j+2)(2j+1)}}\; m \dfrac{\partial}{\partial\theta}\dfrac{Y_{jm}}{\sin\theta} & (\ell = j+1) \\[2ex] -\sqrt{\dfrac{1}{(j+1)(j+2)(2j+1)(2j+3)}}\left[\dfrac{\partial^2}{\partial\theta^2} + (j+1)^2\right] Y_{jm} & (\ell = j+2) \end{cases}$$

5E. $\underset{\sim}{T}{}^{e}_{jm}$ and $\underset{\sim}{T}{}^{m}_{jm}$

The format is similar to that of the preceding section. Since these tensors are transverse, as well as symmetric and traceless, each can be written as a linear combination of the two tensors $\hat{\theta}\hat{\theta} - \hat{\varphi}\hat{\varphi}$ and $\hat{\theta}\hat{\varphi} + \hat{\varphi}\hat{\theta}$. At the end of this section we give explicit expressions for the individual components. Note the symmetries

$$\underset{\sim}{T}{}^{e}_{j-m} = (-1)^{m} \underset{\sim}{T}{}^{e\,*}_{jm} \quad , \qquad \underset{\sim}{T}{}^{m}_{j-m} = (-1)^{m+1} \underset{\sim}{T}{}^{m\,*}_{jm}$$

			$\underset{\sim}{T}{}^{e}_{jm}$		$\underset{\sim}{T}{}^{m}_{jm}$	
j	m	Factor	$\hat{\theta}\hat{\theta} - \hat{\varphi}\hat{\varphi}$	$\hat{\theta}\hat{\varphi} + \hat{\varphi}\hat{\theta}$	$\hat{\theta}\hat{\theta} - \hat{\varphi}\hat{\varphi}$	$\hat{\theta}\hat{\varphi} + \hat{\varphi}\hat{\theta}$
2	2	$\sqrt{\frac{5}{128\pi}}\, e^{2i\varphi}$	$1+\cos^2\theta$	$2i\cos\theta$	$-2\cos\theta$	$-i(1+\cos^2\theta)$
2	1	$\sqrt{\frac{5}{32\pi}}\sin\theta\, e^{i\varphi}$	$\cos\theta$	i	-1	$-i\cos\theta$
2	0	$\sqrt{\frac{15}{64\pi}}\sin^2\theta$	1	0	0	$-i$
2	-1	$\sqrt{\frac{5}{32\pi}}\sin\theta\, e^{-i\varphi}$	$-\cos\theta$	i	-1	$i\cos\theta$
2	-2	$\sqrt{\frac{5}{128\pi}}\, e^{-2i\varphi}$	$1+\cos^2\theta$	$-2i\cos\theta$	$2\cos\theta$	$-i(1+\cos^2\theta)$
3	3	$\sqrt{\frac{21}{256\pi}}\sin\theta\, e^{3i\varphi}$	$-(1+\cos^2\theta)$	$-2i\cos\theta$	$2\cos\theta$	$i(1+\cos^2\theta)$
3	2	$\sqrt{\frac{7}{128\pi}}\, e^{2i\varphi}$	$\cos\theta(3\cos^2\theta-1)$	$2i(2\cos^2\theta-1)$	$2(1-2\cos^2\theta)$	$i\cos\theta(1-3\cos^2\theta)$
3	1	$\sqrt{\frac{35}{256\pi}}\sin\theta\, e^{i\varphi}$	$3\cos^2\theta-1$	$2i\cos\theta$	$-2\cos\theta$	$i(1-3\cos^2\theta)$
3	0	$\sqrt{\frac{105}{64\pi}}\sin^2\theta$	$\cos\theta$	0	0	$-i\cos\theta$
3	-1	$\sqrt{\frac{35}{256\pi}}\sin\theta\, e^{-i\varphi}$	$1-3\cos^2\theta$	$2i\cos\theta$	$-2\cos\theta$	$i(3\cos^2\theta-1)$

j	m	Factor	$\underset{\sim}{T}^{e}_{jm}$: $\hat\theta\hat\theta - \hat\varphi\hat\varphi$	$\underset{\sim}{T}^{e}_{jm}$: $\hat\theta\hat\varphi + \hat\varphi\hat\theta$	$\underset{\sim}{T}^{m}_{jm}$: $\hat\theta\hat\theta - \hat\varphi\hat\varphi$	$\underset{\sim}{T}^{m}_{jm}$: $\hat\theta\hat\varphi - \hat\varphi\hat\theta$
3	-2	$\sqrt{\frac{7}{128\pi}}\, e^{-2i\varphi}$	$\cos\theta(3\cos^2\theta-1)$	$2i(1-2\cos^2\theta)$	$2(2\cos^2\theta-1)$	$i\cos\theta(1-3\cos^2\theta)$
3	-3	$\sqrt{\frac{21}{256\pi}}\sin\theta\, e^{-3i\varphi}$	$1+\cos^2\theta$	$-2i\cos\theta$	$2\cos\theta$	$-i(1+\cos^2\theta)$
4	4	$\sqrt{\frac{63}{512\pi}}\sin^2\theta e^{4i\varphi}$	$1+\cos^2\theta$	$2i\cos\theta$	$-2\cos\theta$	$-i(1+\cos^2\theta)$
4	3	$\sqrt{\frac{63}{256\pi}}\sin\theta e^{3i\varphi}$	$-2\cos^3\theta$	$i(1-3\cos^2\theta)$	$3\cos^2\theta-1$	$2i\cos^3\theta$
4	2	$\sqrt{\frac{9}{128\pi}}e^{2i\varphi}$	$7\cos^4\theta-6\cos^2\theta+1$	$i\cos\theta(7\cos^2\theta-5)$	$\cos\theta(5-7\cos^2\theta)$	$i(-7\cos^4\theta+6\cos^2\theta-1)$
4	1	$\sqrt{\frac{9}{256\pi}}\sin\theta e^{i\varphi}$	$2\cos\theta(7\cos^2\theta-4)$	$i(7\cos^2\theta-1)$	$1-7\cos^2\theta$	$2i\cos\theta(4-7\cos^2\theta)$
4	0	$\sqrt{\frac{45}{256\pi}}\sin^2\theta$	$7\cos^2\theta-1$	0	0	$i(1-7\cos^2\theta)$
4	-1	$\sqrt{\frac{9}{256\pi}}\sin\theta e^{-i\varphi}$	$2\cos\theta(4-7\cos^2\theta)$	$i(7\cos^2\theta-1)$	$1-7\cos^2\theta$	$2i\cos\theta(7\cos^2\theta-4)$
4	-2	$\sqrt{\frac{9}{128\pi}}\, e^{-2i\varphi}$	$7\cos^4\theta-6\cos^2\theta+1$	$i\cos\theta(5-7\cos^2\theta)$	$\cos\theta(7\cos^2\theta-5)$	$i(-7\cos^4\theta+6\cos^2\theta-1)$
4	-3	$\sqrt{\frac{63}{256\pi}}\sin\theta e^{-3i\varphi}$	$2\cos^3\theta$	$i(1-3\cos^2\theta)$	$3\cos^2\theta-1$	$-2i\cos^3\theta$
4	-4	$\sqrt{\frac{63}{512\pi}}\sin^2\theta e^{-4i\varphi}$	$1+\cos^2\theta$	$-2i\cos\theta$	$2\cos\theta$	$-i(1+\cos^2\theta)$

Explicit expressions for the components of $\underset{\sim}{T}^{e}_{jm}$ and $\underset{\sim}{T}^{m}_{jm}$:

$$\underset{\sim}{T}^{e}_{jm}(\Omega) = (\hat\theta\hat\theta - \hat\varphi\hat\varphi)\sqrt{\frac{1}{2j(j+1)(j-1)(j+2)}}\left[2\frac{\partial^2}{\partial\theta^2} + j(j+1)\right]Y_{jm}$$

$$+ (\hat\theta\hat\varphi + \hat\varphi\hat\theta)\sqrt{\frac{2}{j(j+1)(j-1)(j+2)}}\; i\,m\,\frac{\partial}{\partial\theta}\left(\frac{Y_{jm}}{\sin\theta}\right)$$

$$\underset{\sim}{T}{}^{m}_{jm}(\Omega) = -(\hat{\theta}\hat{\theta} - \hat{\varphi}\hat{\varphi}) \sqrt{\frac{2}{j(j+1)(j-1)(j+2)}}\; m \frac{\partial}{\partial \theta} \left(\frac{Y_{jm}}{\sin\theta} \right)$$

$$-(\hat{\theta}\hat{\varphi} + \hat{\varphi}\hat{\theta}) \sqrt{\frac{1}{2j(j+1)(j-1)(j+2)}}\; i \left[2 \frac{\partial^2}{\partial \theta^2} + j(j+1) \right] Y_{jm}$$

Addenda

$$\left[\frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + J(J+1)\sin\theta - \frac{M^2}{\sin\theta} \right] Y_{JM} = 0$$

$\sin\theta\ \hat{\varphi}\times\vec{Y}_{JLM}$

$\underline{L=J+1}$:

$$\frac{-1}{2J+1}\sqrt{\frac{(J^2-M^2)(J+1)}{J}}\ \vec{Y}_{J-1\,J\,M}$$

$$+\ \frac{M}{\sqrt{J(2J+1)}}\ \vec{Y}_{JJM}$$

$$+\ \frac{2}{(2J+3)(2J+1)}\sqrt{(J+1)^2-M^2}\ \vec{Y}_{J+1\,J\,M}$$

$$+\ \frac{1}{2J+3}\sqrt{\frac{[(J+1)^2-M^2](J+2)}{J+1}}\ \vec{Y}_{J+1\,J+2\,M}$$

$\underline{L=J}$:

$$\frac{M}{\sqrt{(J+1)(2J+1)}}\ \vec{Y}_{J\,J-1\,M}\ -\ \frac{M}{\sqrt{J(2J+1)}}\ \vec{Y}_{J\,J+1\,M}$$

$\underline{L=J-1}$:

$$\frac{1}{2J-1}\sqrt{\frac{(J^2-M^2)(J-1)}{J}}\ \vec{Y}_{J-1\,J-2\,M}$$

$$-\ \frac{2}{(2J+1)(2J-1)}\sqrt{J^2-M^2}\ \vec{Y}_{J-1\,J\,M}$$

$$-\ \frac{M}{\sqrt{(J+1)(2J+1)}}\ \vec{Y}_{JJM}$$

$$-\ \frac{1}{2J+1}\sqrt{\frac{[(J+1)^2-M^2]\,J}{J+1}}\ \vec{Y}_{J+1\,J\,M}$$